CONGESTION IN A SYSTEM OF N SERVICE CENTERS WITH INTERFERENCE

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NAVAL POSTGRADUATE SCHOOL

Monterey, California

THESIS

CONGESTION IN A SYSTEM OF N

SERVICE CENTERS WITH INTERFERENCE

by

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We have a system of N sendox machines (service centers) which can transmit or receive (but not simultaneously) photocopies to and from each other. The stochastic process $\{X_M(t),\ t \geq 0\}$, where $X_M(t)$ is the number of messages undergoing transmission in the system at epoch t, is selected to be the measure of system congestion. Under the assumptions

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of exponential inter-arrival and transmission times and that messages not transmitted immediately are lost, $X_M(t)$ is a birth and death process. We call the model under these assumptions the "loss" model. It is shown that $X_M(t)$ can be approximated by a diffusion process. The steady state results of the diffusion model can be calculated with ease and are in close agreement with those of the birth and death model and the simulation "loss" model. A more realistic "no loss" model is also developed where messages that are not transmitted immediately are allowed to queue up. It is shown, however, that this "no loss" model can be quite accurately approximated by an appropriate "loss" model.

Congestion in a System of N

Service Centers With Interference

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We have a system of N sendox machines (service centers) which can transmit or receive (but not simultaneously) photocopies to and from each other. The stochastic process $\{X_M(t), t \geq 0\}$, where $X_M(t)$ is the number of messages undergoing transmission in the system at epoch t, is selected to be the measure of system congestion. Under the assumptions of exponential inter-arrival and transmission times and that messages not transmitted immediately are lost, $X_M(t)$ is a birth and death process. We call the model under these assumptions the "loss" model. It is shown that $X_M(t)$ can be approximated by a diffusion process. The steady state results of the diffusion model can be calculated with ease and are in close agreement with those of the birth and death model and the simulation "loss" model. A more realistic "no loss" model is also developed where messages that are not transmitted immediately are allowed to queue up. It is shown, however, that this "no loss" model can be quite accurately approximated by an appropriate "loss" model.

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NOTATIONS

N	total number of machines (service centers) in the system. (Assumed to be an even number.)
$M=N/2$	maximum number of messages that can be transmitted by the system.
λ	arrival rate of messages at each machine.
λ_{eff}	effective arrival rate of messages at each machine in the "loss model.
μ	departure rate of a message undergoing transmission.
ρ	$\frac{\mu}{2\lambda}$
$X_M(t)$	number of messages undergoing transmission at epoch t.
$S_M(t)$	stochastic element associated with $X_M(t)$.
$y(t)$	deterministic component of $X_M(t)$.
$\phi_M(t)$	characteristic function of $X_M(t)$.
$\psi_M(t)$	characteristic function of $S_M(t)$.
x_s	steady state mean of $X_M(t)$.
σ_s^2	steady state variance of $X_M(t)$.
P_j's	steady state probabilities of $X_M(t)$.

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I. INTRODUCTION

The main purpose of this thesis is to construct and compare various models which can be used to analyze the degree of congestion in a system of N sendox facsimile machines. Sendox machines can both transmit and receive (but not simultaneously) photocopies using standard telephone lines. The original problem arose out of a proposal to install a number of these machines in central London Mintech buildings for transmission of thin "immediate" messages between buildings. This problem was first studied by Coleman [Ref. 1] who investigated the expected system congestion, expected cost per message and a possible priority system.

One of the models in his study considered each machine individually as a single server system. It was assumed that only one machine was located in each building. Arrivals at each of the N machines occur from two sources: 1) requests to transmit messages to other machines and 2) incoming messages from the other N-1 machines. These two arrival streams were merged into one and the standard queuing formula for a single server queue was applied to obtain the expected waiting time of messages at an individual machine. However, a subsequent simulation study showed wide discrepancy between the theoretical and simulated expected waiting times. This occurred because the single server model did not take into account the "interference' between machines. The model assumed that when a message is transmitted from one machine,

its destination (which could be any one of the other N-1 machines) is always available to receive the message. In reality, its destination may be engaged in transmitting or receiving another message. The effect of such "interference" tends to increase the waiting time as compared to the result obtained from the single server queuing model. Any reasonable theoretical model, therefore, has to consider all N machines simultaneously. It is very difficult, however, to build an analytical model that can give a reasonable estimate of waiting time because it has to consider N queues all at once.

II. DESCRIPTION OF THE "LOSS" MODEL

The problem becomes much more tractable when we only consider modelling the system size i.e., the number of messages undergoing transmission. This number varies from 0 to M where M=[N/2], the largest integer less than or equal to N/2. Let $X_M(t)$ be defined as the number of messages undergoing transmission at epoch t. The stochastic process $\{X_M(t), t \geq 0\}$ will be our measure of system congestion.

A number of assumptions are made to facilitate construction of theoretical models for the process $X_M(t)$. We assume a Poisson arrival of messages at each machine. The arrival rate of messages requesting transmission at each machine is the same and is denoted by λ. Moreover, we assume that each message undergoing transmission is finishing at an exponential rate of μ. Furthermore we define ρ to be $\frac{\mu}{2\lambda}$ and hence the departure rate (μ) of a message is also equal to $2\rho\lambda$. Finally, we assume that if a message fails to be transmitted immediately upon arrival (due to either the receiving or sending machine being busy), then the message is lost to the system. For this reason, we call the model based on the above assumptions the "loss" model.

Two theoretical and two simulation models for the process $X_M(t)$ will be described and discussed in the following sections. We are mainly interested in steady state behavior of the process $X_M(t)$. In the first theoretical model, $X_M(t)$ is shown to be a birth and death process (this was done in

[Ref. 1]) whose steady state probabilities can be derived and calculated numerically. Since Iglehart [Ref. 3] had proved the convergence of certain birth and death processes to diffusion processes, we were led to consider a diffusion approximation for the process $X_M(t)$, when M is large. This was the approach taken in the second theoretical model where it was shown that $X_M(t)$ can be approximated by a non-stationary Ornstein-Uhlenbeck process. Finally, a simulation model was built to validate the theoretical models. The steady state results of the theoretical and the simulation models compared very favorably with one another.

A second simulation model was developed for the case in which messages not transmitted immediately, queue up until they are finally transmitted. We call this model the "no loss" model. We are unable to construct a theoretical model for this case. However we will show that this "no loss" model can be quite accurately approximated by the diffusion "loss" model.

III. BIRTH AND DEATH "LOSS" MODEL

In an unpublished paper [Ref. 1], Dr. Rodney Coleman modelled the number of messages undergoing transmission, $X_M(t)$, as a birth and death process. For the sake of simplicity, we assume that N, the total number of machines in the system is even and hence M, the maximum number of messages that can be carried by the system is given by N/2. When the process $X_M(t)$ is in state j, the conditional probability that $X_M(t)$ will visit state j+1 in the next interval of length Δt is $\lambda_j \Delta t + o(\Delta t)$ and the conditional probability that $X_M(t)$ will visit state j-1 is $\mu_j \Delta t + o(\Delta t)$, where λ_j and μ_j are the infinitesimal birth and death rates respectively [Ref. 6]. Messages can only enter the system at the (2M-2j) free machines at a rate of $(2M-2j)\lambda$. When a message arrives at a free machine, its probability of finding an available receiving machine is $\frac{2M-2j-1}{2M-1}$, therefore,

$$\lambda_j = \frac{(2M-2j)(2M-2j-1)\lambda}{(2M-1)} \tag{3.1}$$

Since there are j messages, each finishing transmission at the rate of μ, and thus

$$\mu_j = j\mu = 2j\rho\lambda \tag{3.2}$$

Let P_j denote the steady state probability of $X_M(t)$ being in state j. Then Coleman [Ref. 1] showed that

$$P_k = \frac{\dfrac{\gamma^k}{k!(2M-2k)!}}{\sum_{k=0}^{M} \dfrac{\gamma^k}{k!(2M-2k)!}} \qquad k=0,1,\ldots M$$

where $\gamma = (\frac{1}{2M-1}) \frac{\lambda}{\mu} = \frac{1}{2\rho(2M-1)}$

However, since the caluclation for the P_k's are carried out on a digital computer, a more efficient method computationally is to use the following recursive relationships [Ref. 6].

$$\mu_k P_k = \lambda_{k-1} P_{k-1} \qquad k=1,2,\ldots M \tag{3.3}$$

$$\sum_{k=0}^{M} P_k = 1 \tag{3.4}$$

The computer program which calculates the steady state probabilities and also the mean and variance of $X_M(t)$ at steady state is listed in Appendix D.

IV. DIFFUSION "LOSS" MODEL

We follow Gaver's [Ref. 4] approach by approximating the process $X_M(t)$ with a deterministic component $y(t)$ and a stochastic element or noise $S_M(t)$. Thus $X_M(t)$ can be written as:

$$X_M(t) = My(t) + \sqrt{M}\, S_M(t) \tag{4.1}$$

We show that as we let M go to infinity, $S_M(t)$ converges to a non-stationary Ornstein-Uhlenbeck process, $S(t)$ [Ref. 2]. Hence for large M, we can use the following formula to approximate $X_M(t)$:

$$X_M(t) \simeq My(t) + \sqrt{M}\, S(t) \tag{4.2}$$

Because of our assumption of Poisson arrivals, a message will arrive at a machine with probability $\lambda dt + o(dt)$ in the time interval $(t,t+dt)$. During this time interval, the probability of two or more messages arriving at the same machine is $o(dt)$. At epoch t the number of messages undergoing transmission is $X_M(t)$ and hence the number of unoccupied machines is given by $N-2X_M(t)$ or $2M-2X_M(t)$. In the time interval $(t,t+dt)$, effective arrival of messages can occur only at any of the $2M-2X_M(t)$ unoccupied machines. Given that a message has arrived at an available machine, its probability of finding the receiving machine not in use is now $(2M-2X_M(t)-1)/(2M-1)$. Therefore,

$$\text{Prob}\{X_M(t+dt)=X_M(t)+1 \mid \text{given } X_M(t)\}$$
$$= \frac{(2M-2X_M(t))(2M-2X_M(t)-1)\lambda dt}{2M-1} + o(dt) \tag{4.3}$$

Similarly, under the assumption of exponential transmission times, each message undergoing transmission is finishing at the rate of μ or $2\rho\lambda$. At epoch t, messages will be finishing at a rate of $2X_M(t)\rho\lambda$. Therefore during the time interval $(t,t+dt)$ the probability of a message completing its transmission is $2X_M(t)\rho\lambda dt + o(dt)$ and the probability of two or more messages completing their transmission is $o(dt)$. Therefore,

$$\begin{aligned}&\text{Prob }\{X_M(t+dt) = X_M(t)-1 \mid \text{given } X_M(t)\}\\ &= 2X_M\rho\lambda dt + o(dt)\end{aligned} \tag{4.4}$$

Finally,

$$\begin{aligned}&\text{Prob }\{X_M(t+dt) = X_M(t) \mid \text{given } X_M(t)\}\\ &= 1 - \frac{(2M-2X_M(t))(2M-2X_M(t)-1)}{2M-1}\lambda dt - 2X_M(t)\rho\lambda dt + o(dt)\end{aligned} \tag{4.5}$$

In Appendix A, equations (4.1) through (4.5) are used to derive the partial differential equation

$$\frac{\partial\psi}{\partial t} = -2\lambda\{2+\rho-2y(t)\}\theta\frac{\partial\psi}{\partial\theta} - 2\lambda\{y^2(t)-y(t)(2-\rho)+1\}\frac{\theta^2}{2}\psi \tag{4.6}$$

Here $\psi(\theta,t)$ is the characteristic function of $S(t)$, the limit of the noise process $S_M(t)$ as $M \to \infty$. The deterministic process $y(t)$ must satisfy the ordinary differential equation

$$y'(t) - 2\lambda\{y^2(t) - (2+\rho)y(t) - 1\} = 0 \tag{4.7}$$

in order that the partial differential equation (4.6) be valid. The partial differential equation (4.6) is recognized as the transformed version of the forward differential

equation for a non-stationary Ornstein-Uhlenbeck process. It is known [Ref. 2] that the probability density function of the Ornstein-Uhlenbeck process S(t) must satisfy the Kolmogorov Forward Differential Equation

$$\frac{\partial p}{\partial t} = \tfrac{1}{2} \frac{\partial^2}{\partial s^2} \{\alpha(s,t)p\} - \frac{\partial}{\partial s} \{\beta(s,t)p\} \tag{4.8}$$

where $\beta(s,t)$ and $\alpha(s,t)$ denote the infinitesimal mean and variance respectively, conditioned on $S(t)=s$.

$$\beta(s,t) = -2\lambda\{2+\rho-2y(t)\}s \tag{4.9}$$

$$\alpha(s,t) = 2\lambda\{y^2(t)-(2-\rho)y(t)+1\} \tag{4.10}$$

The first step toward determining the mean and variance of the process S(t) involves solving the first order differential equation (4.7) to obtain y(t) explicitly, assuming the initial condition that $y(0)=0$. The solution is derived in Appendix A and is

$$y(t) = v-w \left\{\frac{1 + ke^{-4\lambda wt}}{1 - ke^{-4\lambda wt}}\right\} \tag{4.11}$$

where $v = 1 + \tfrac{1}{2}\rho$

$w^2 = (1 + \tfrac{1}{2}\rho)^2 - 1$

$k = \frac{w-v}{w+v}$

Since the Ornstein-Uhlenbeck process is Gaussian, we may differentiate the Gaussian characteristic function $\exp\ [i\theta a(t)-\tfrac{1}{2}\theta^2 b(t)]$ with respect to θ and t, substitute ψ, $\frac{\partial\psi}{\partial\theta}$, $\frac{\partial\psi}{\partial t}$ back into equation (4.6) and equate the coefficients of $i\theta$ and θ^2 in order to obtain two first order differential equations for a(t) and b(t) which denote the mean and

variance of the process S(t) respectively. Assuming that S(0)=0, we can show that

$$E[S(t)] = 0 \qquad t \geq 0 \tag{4.12}$$

and since

$$X_M(t) \simeq My(t) + \sqrt{M}\, S(t)$$

we obtain

$$E[X_M(t)] \simeq My(t) \qquad t \geq 0 \tag{4.13}$$

where y(t) is given by equation (4.11)

The solution of the first order differential equation for the Var {S(t)} involves integrals that are quite intractable and we are only able to obtain an approximation for large t.

$$\text{Var } \{S(t)\} \quad \frac{y_e^2 - y_e(2-\rho)+1}{2(2+\rho-2y_e)} \qquad t \text{ large} \tag{4.14}$$

where $$y_e = \lim_{t\to\infty} y(t) = 1 + \tfrac{1}{2}\rho - \sqrt{(1+\tfrac{1}{2}\rho)^2-1} \tag{4.15}$$

Since Var $\{S(t)\} \simeq \frac{1}{M}$ Var $\{X_M(t)\}$, we obtain the following expressions for steady state mean and variance of $X_M(t)$, denoted by x_s and σ_s^2 respectively.

$$\begin{aligned} x_s &= E[X_M(t)] = My_e \qquad t \text{ large} \\ \sigma_s^2 &= \text{Var } \{X_M(t)\} \end{aligned} \tag{4.16}$$

$$= M \left[\frac{y_e^2-y_e(2-\rho)+1}{2(2+\rho-2y_e)}\right] \qquad t \text{ large} \tag{4.17}$$

Detailed derivation of equations (4.12) and (4.14) is given in Appendix C. When t is large, $X_M(t)$ is normally distributed with mean and variance given by equations (4.16) and

(4.17) respectively. Plots of x_s/M and σ_s^2/M against ρ are given in Figures 1 and 2 respectively. It is seen in Appendix C that y(t) reaches its steady state value of y_e considerably faster than Var {S(t)}, thus the steady state mean of $X_M(t)$ is attained much faster than the steady state variance of $X_M(t)$.

The steady state probabilities for $X_M(t)$ can be easily obtained from the standard normal table after x_s and σ_s^2 are calculated using equations (4.16) and (4.17). However, since we are using a continuous (normal) probability distribution to approximate a discrete distribution, we have to make the following "continuity corrections"

$$P_o = \int_{-\infty}^{\frac{1}{2}} \frac{1}{\sqrt{2\pi}\sigma_s} e^{-\frac{(x-x_s)^2}{2\sigma_s^2}} dx \tag{4.18}$$

$$P_i = \int_{i-\frac{1}{2}}^{i+\frac{1}{2}} \frac{1}{\sqrt{2\pi}\sigma_s} e^{-\frac{(x-x_s)^2}{2\sigma_s^2}} dx \qquad i=1,2,\ldots M-1 \tag{4.19}$$

$$P_M = \int_{M-\frac{1}{2}}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_s} e^{-\frac{(x-x_s)^2}{2\sigma_s^2}} dx \tag{4.20}$$

Another way to derive the parameters for the Ornstein-Uhlenbeck process S(t) is by defining the stochastic differential equation for the process $X_M(t)$.

$$dX_M(t) = \beta_M(x,t)dt + Z(t)\sqrt{\alpha_M(x,t)dt} \tag{4.21}$$

where Z(t) is a purely random Gaussian process with zero mean and unit variance and $\beta_M(x,t)dt$ and $\alpha_M(x,t)dt$ are the mean and variance of the increment $dX_M(t)$ in a small time

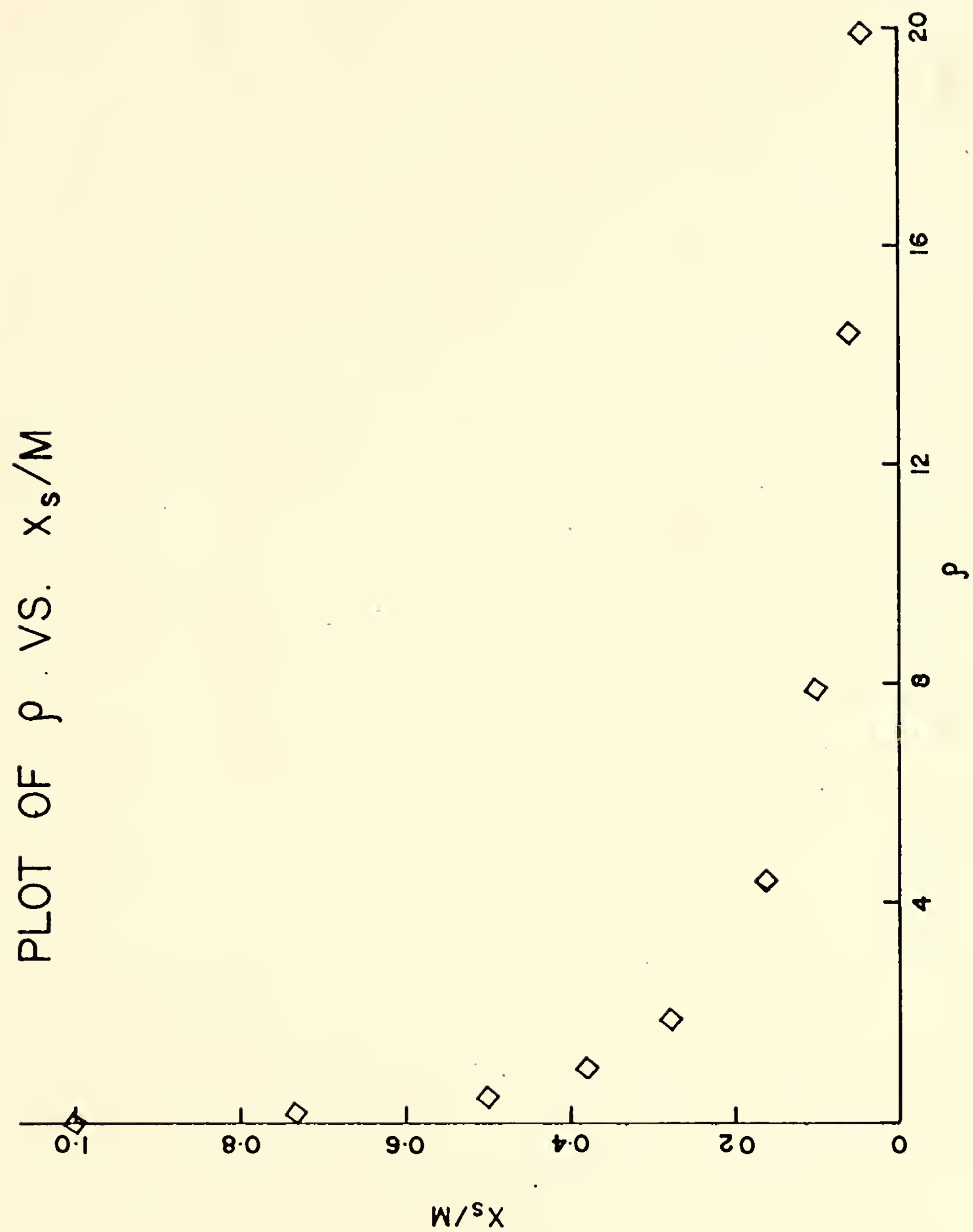


Figure 1

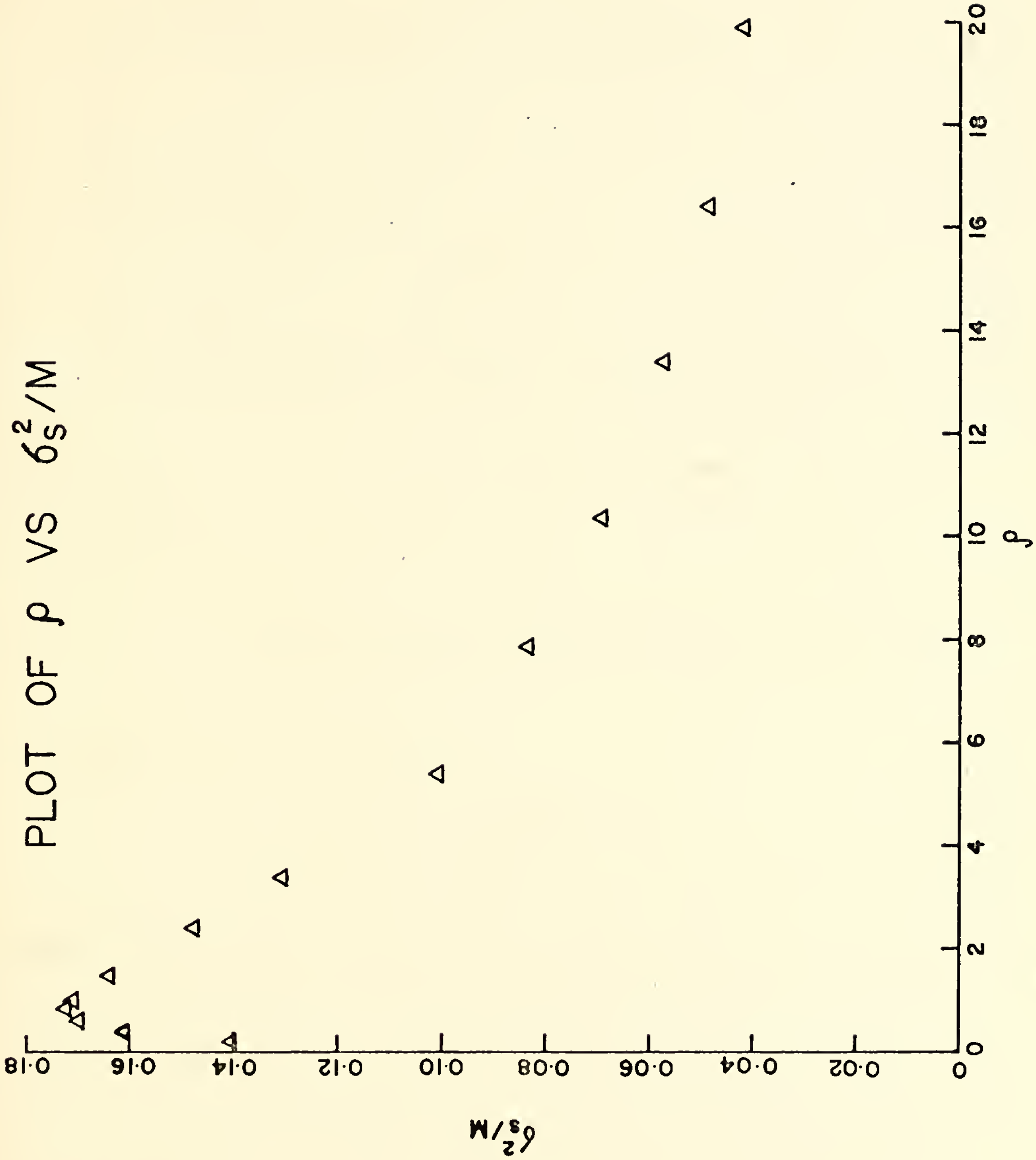

PLOT OF ρ VS σ_s^2/M
σ_s^2/M
ρ
0
0.02
0.04
0.06
0.08
0.10
0.12
0.14
0.16
0.18
2
4
6
8
10
12
14
16
18
20

Figure 2

interval $(t,t+dt)$. Recall that we have derived the following probabilities earlier

$$\text{Prob } \{dX_M(t) = 1 \mid \text{ given } X_M(t)\}$$

$$= \frac{\{2M-2X_M(t)\}\{2M-2X_M(t)-1\}}{2M-1} \lambda dt + o(dt)$$

$$\text{Prob } \{dX_M(t) = -1 \mid \text{ given } X_M(t)\}$$

$$= 2X_M(t)\rho\lambda dt + o(dt)$$

$$\text{Prob } \{dX_M(t) = 0 \mid \text{ given } X_M(t)\}$$

$$= 1 - \frac{\{2M-2X_M(t)\}\{2M-2X_M(t)-1\}}{2M-1} \lambda dt - 2X_M(t)\rho\lambda dt + o(dt)$$

Therefore it is reasonable to define $\beta_M(x,t)dt$ and $\alpha_M(x,t)dt$ as follows:

$$\begin{aligned} \beta_M(x,t)dt &= E[dX_M(t)] \\ &= \frac{\{2M-2X_M(t)\}\{2M-2X_M(t)-1\}}{2M-1} \lambda dt - 2X_M(t)\rho\lambda dt \end{aligned} \tag{4.22}$$

$$\begin{aligned} \alpha_M(x,t)dt &= \text{Var } [dX_M(t)] \\ &= \frac{\{2M-2X_M(t)\}\{2M-2X_M(t)-1\}}{2M-1} \lambda dt + 2X_M(t)\rho\lambda dt \end{aligned} \tag{4.23}$$

Substituting $S_M(t)$ as given by equation (4.1) in place of $X_M(t)$, we derive a stochastic differential equation for $S_M(t)$ in Appendix B. Then by letting $M \to \infty$ and requiring the first order differential equation (4.7) for the deterministic process $y(t)$ to hold, we obtain the stochastic differential equation

$$dS(t) = 2\lambda\{2y(t)-2-\rho\}S(t)dt$$
$$+ Z(t)\sqrt{2\lambda\{y^2(t)-y(t)(2-\rho)+1\}dt} \tag{4.24}$$

This equation is the stochastic differential equation for an Ornstein-Uhlenbeck process which has the same infinitesimal mean and variance as those given in equations (4.9) and (4.10).

V. SIMULATION "LOSS" MODEL

A. SIMULATION PROGRAM I

Two methods of simulations are presented here for the "loss" model. Simulation Program I makes use of the obvious approach of duplicating the operation of the actual system. Messages arrive at each of the N machines at a Poisson rate λ. These messages are either lost or undergo transmissions which are being completed at an individual rate of μ. A brief description of the program logic is given in the next paragraph.

Simulation Program I is written in FORTRAN. It operates simply by searching for the next event to occur. In this model an event is either the arrival of a message at a machine or the completion of a message undergoing transmission. Each event is assigned a clock time which denotes the time of occurrence of that event. The next event is that whose clock time is the minimum of the clock times of all the future events. Having found this event, the program performs a series of tests to determine whether the conditions in the system are such that the potential event can take place. If so, the program moves to execute this event. For example the event may be the arrival of a message at machine i, then the necessary test is to check whether the sending machine i and a randomly selected receiving machine j, $j \neq i$ are both available. If so, the number of messages undergoing transmission increases by one. The basic loop is repeated as

many times as necessary to complete the entire simulation. The flow chart and listing of this program are given in Appendix D.

B. SIMULATION PROGRAM II

Another way to develop a simulation program for the "loss" model is to make use of the fact that $X_M(t)$ is a birth and death process. We know the process $X_M(t)$ stays in state i for an amount of time that is exponential with rate $\lambda_i+\mu_i$ and given that it makes a transition from state i, it goes to either state i-1 or i+1 with probabilities $\frac{\mu_i}{\lambda_i+\mu_i}$ and $\frac{\lambda_i}{\lambda_i+\mu_i}$ respectively where λ_i and μ_i are as defined in section III.

Instead of monitoring the flow of messages through the system of N machines, Program II works by keeping track of the length of time that the process $X_M(t)$ spends in each state and when $X_M(t)$ changes state. This program is also written in FORTRAN. It starts off with $X_M(0)$ in a given initial state i. An exponential variate with mean $\frac{1}{\lambda_i+\mu_i}$ is generated and this represents the length of time that $X_M(t)$ spends in state i before jumping to state i-1 or i+1. In order to decide whether $X_M(t)$ next visits i-1 or i+1, a uniform random variate p, $0 \leq p \leq 1$, is generated. If p is less than $\frac{\lambda_i}{\lambda_i+\mu_i}$, then $X_M(t)$ visits state i+1 next and if p exceeds $\frac{\lambda_i}{\lambda_i+\mu_i}$ $X_M(t)$ goes into state i-1. Once $X_M(t)$ switches to a new state j, we repeat the process of determining the length of $X_M(t)$'s stay in state j and the next state to be

visited. If $X_M(t)$ is in state 0, it will next visit state 1 with probability one. Similarly if $X_M(t)$ is in state M, it will next visit state M-1 with probability one. The output of this program will be discussed together with that of Program I in the next section. The flow chart and listing of Program II are included in Appendix D.

C. DISCUSSION OF OUTPUT OF PROGRAMS I AND II

Both simulation programs have five input parameters, namely the total number of machines (N), the duration of each simulation run (TSTOP), the arrival rate of messages at each machine (λ), the departure rate of transmitted messages (μ) and the initial state i. Since we are mainly interested in steady state results, the primary output of both simulation programs is the probability distribution of the process $X_M(t)$ when in steady state.

One way to obtain these steady state probabilities is to let the simulation program run for t' minutes until we are reasonably sure that steady state has been achieved, and then record that value of $X_M(t')$. Since each simulation run produces only one realization of $X_M(t')$, this method of estimating the P_j's proves to be very inefficient because of the large number of simulation runs required to produce a consistent estimate.

A better method to estimate the P_j's is to observe the time average probabilities of $X_M(t)$ when it is already in steady state. We choose a starting state i which is very close to the steady state mean My_e so that $X_M(t)$ will reach

steady state shortly. The simulation model will have to run for a considerable amount of time in order to gather enough sample values. However, this results in a program that requires too much core storage. To overcome this problem, we made 200 simulation runs of TSTOP minutes instead of a single run of 200 × TSTOP minutes.

During each execution of the simulation program, 200 simulation runs of duration TSTOP minutes each are made. Let f_i denote the frequency that the process $X_M(t)$ visits state i. At the beginning of the program, all the f_i's are set equal to zero. During each simulation run, the number of messages undergoing transmission is observed at discrete one minute intervals. If this number is k, then f_k is incremented by one and all other f_j's, $j \neq k$, remain unchanged. Thus, during the entire simulation program, $X_M(t)$ is observed at 200 × TSTOP points, and the estimated steady state probabilities are given by:

$$\hat{p}_k = \frac{f_k}{200 \times \mathrm{TSTOP}}$$

Initially, both programs are tested with the same input parameters to ensure that they are yielding results that are consistent with each other. From then on, only Program II is used to generate the various output that are used to validate the theoretical models, because time-wise, it is much more efficient than Program I. For example, with N=50, λ=2.5 and μ=10.0, the total execution time of Program I is about 18 times that of Program II.

VI. SIMULATION "NO LOSS" MODEL

In the previous sections, we have developed analytical and simulation models for a system of N machines in which messages that are not transmitted immediately are lost. A more realistic assumption is to allow these messages that are not transmitted immediately to join a queue at the respective machines. We call a model based on this less stringent assumption a "no loss" model. This "no loss" model is not easily amenable to theoretical analysis, but it can be easily simulated by a program written in GPSS. The assumptions of exponential distribution for the inter-arrival and inter-departure times of messages could now be relaxed because a simulation program can easily handle any other probability distribution. However, the previous assumption of exponential inter-arrival and transmission times is retained for comparison purposes.

In the GPSS program, the Sendox machines are treated as facilities and the messages that arrive at the various machines are treated as transactions. The program consists mainly of N closely similar segments. Each segment simulates the activity of a facility. The program instructions for each segment are almost identical except for the fact that each facility has a different facility number i, i=1,2,...N.

Arrivals of messages are simulated through the generation of random arrivals of transactions at an exponential rate λ at each facility. A transaction created at facility i is

randomly assigned to a destination facility j, $j \neq i$, independently of other transactions. Then facilities i and j are tested to see whether they are available simultaneously. If so, the transaction proceeds to seize both facilities for a length of time that is exponentially distributed with rate μ. If not, the transaction will have to wait until both facilities i and j are simultaneously free. If another transaction is created while a previous transaction is either waiting for or undergoing service, the new transaction must queue up. Therefore we assume a first come first serve queue discipline. This procedure is followed at all N facilities.

GPSS is preferred to FORTRAN in this model because the logic and structure of a GPSS program are more simple. Moreover queue statistics and steady state probabilities of $X_M(t)$ can be gathered with little extra effort. The only shortcoming is that this program cannot adapt easily to changes in the number of machines. Thus if we want to double the number of machines the card deck will almost double its size. This GPSS program was only written for N equal to 20, whereas N can go as high as 200 in Programs I and II. A listing of this program is included in Appendix D.

VII. COMPARISON OF RESULTS

According to the diffusion "loss" model, when $X_M(t)$ is in steady state $X_M(t)$ is approximately normally distributed with mean x_s and variance σ_s^2 given by equations (4.16) and (4.17). Moreover, using x_s and σ_s^2 and the continuity correction, given by equations (4.18), (4.19) and (4.20), the steady state probabilities can be obtained from the standard normal table.

We are mainly interested in determining the accuracy of the diffusion "loss" model for various values of N and ρ in comparison with the simulation models. For ease of reference equations (4.16) and (4.17) are repeated here.

$$x_s = My_e = \frac{N}{2}\,y_e \tag{7.1}$$

$$\sigma_s^2 = M\,\frac{y_e^2 - y_e(2-\rho)+1}{2(2+\rho-2y_e)} \tag{7.2}$$

where $\quad y_e = 1 + \frac{1}{2}\rho - \sqrt{(1+\frac{1}{2}\rho)^2 - 1}$

Equations (7.1) and (7.2) implies that x_s and σ_s^2 depends only on the ratio of λ and μ, when M is kept constant. To check this implication, Simulation Program II was run with five sets of λ and μ where the ratio $\frac{\mu}{2\lambda}$ is kept equal to 2. The steady state means and variances obtained from these runs turned out to be extremely close to x_s and σ_s^2 as calculated from equations (7.1) and (7.2) and these results are given in Table I.

<u>Table I</u>

Steady State Mean and Variance of $X_M(t)$ with

ρ constant, N = 100, $\rho = \frac{\mu}{2\lambda} = 2.0$.

Arrival rate of messages λ (# per hour)	Departure rate of messages μ (# per hour)	Simulation "loss" model		Diffusion "loss" model	
		x_s	σ_s^2	x_s	σ_s^2
1.25	5.0	13.389	7.712	13.389	7.723
2.5	10.0	13.406	7.753	13.389	7.723
5.0	20.0	13.418	7.787	13.389	7.723
10.0	40.0	13.410	7.760	13.389	7.723
20.0	80.0	13.408	7.760	13.389	7.723

(Each simulation runs for 2400 hours)

Next, keeping M=10, or N=20, six more runs were made with the values of ρ ranging from 0.5 to 10. Again x_s and σ_s^2 calculated from equations (7.1) and (7.2) were in close agreement with the simulation results. These results are summarized in Table II.

Equations (7.1) and (7.2) also imply that x_s and σ_s^2 are linear functions of N keeping ρ constant at 2. Six simulation runs were made for values of N ranging from 10 to 200. The steady state probabilities for the birth and death "loss" model were also calculated for the same values of ρ and N. Again, the results indicated that all three models agree very well as far as steady state mean, variance and cumulative probabilities are concerned. The steady state means and variances for the three models are given in Tables III and IV respectively. In Table V we compared the cumulative probabilities of the three models for the case where N=20 and ρ=2. These same values when plotted on normal probability graph paper (Figure 3), gave clear visual evidence of their close agreement.

Table II

Steady State Mean and Variance of $X_M(t)$ with

N constant, $N = 40$, $\lambda = 2.5$.

Departure rate of messages μ (# per hour)	$\rho = \frac{\mu}{2\lambda}$	Simulation "loss" model		Diffusion "loss" model	
		x_s	σ_s^2	x_s	σ_s^2
2.5	0.5	9.976	3.647	10.000	3.333
5.0	1.0	7.631	3.544	7.600	3.413
7.5	1.5	6.263	3.357	6.277	3.299
10.0	2.0	5.352	3.117	5.359	3.094
30.0	6.0	2.537	1.969	2.540	1.967
50.0	10.0	1.676	1.421	1.678	1.415

(Each simulation runs for 2400 hours)

Table III

Steady State Mean of $X_M(t)$ with λ and μ constant,

$\lambda = 2.5$, $\mu = 10.0$ and $\rho = \frac{\mu}{2\lambda} = 2.0$.

Number of machines N	Steady State Mean, x_s		
	Simulation "loss" model	Diffusion "loss" model	Birth and Death "loss" model
20	2.703	2.680	2.697
40	5.373	5.359	5.376
60	8.049	8.038	8.055
80	10.722	10.718	10.735
100	13.397	13.398	13.414
200	26.803	26.795	26.811

(Each simulation runs for 2400 hours)

Table IV

Steady State Variance of $X_M(t)$ with λ and μ constant, $\lambda = 2.5$, $\mu = 10.0$ and $\rho = \frac{\mu}{2\lambda} = 2.0$.

Number of machines N	Steady State Variance σ_s^2		
	Simulation "loss" model	Diffusion "loss" model	Birth and Death "loss" model
20	1.547	1.547	1.559
40	3.052	3.094	3.106
60	4.617	4.641	4.653
80	6.142	6.188	6.199
100	7.723	7.735	7.746
200	15.440	15.470	15.484

(Each simulation runs for 2400 hours)

Table V

Steady State Cumulative Probabilities, N=20,

$M=N/2=10,\ \lambda = 2.5,\ \mu = 10.0,\ \rho = \frac{\mu}{2\lambda} = 2.0.$

State j	Steady State Cumulative Probabilities P_j		
	Simulation "loss" model	Diffusion "loss" model	Birth and Death "loss" model
0	0.0275	0.0401	0.0280
1	0.1646	0.1716	0.1681
2	0.4459	0.4427	0.4501
3	0.7485	0.7453	0.7470
4	0.9243	0.9284	0.9247
5	0.9865	0.9887	0.9864
6	0.9985	0.9989	0.9986
7	0.9999	1.0000	0.9999
8	1.0000	1.0000	1.0000
9	1.0000	1.0000	1.0000
10	1.0000	1.0000	1.0000

(Each simulation runs for 2400 hours)

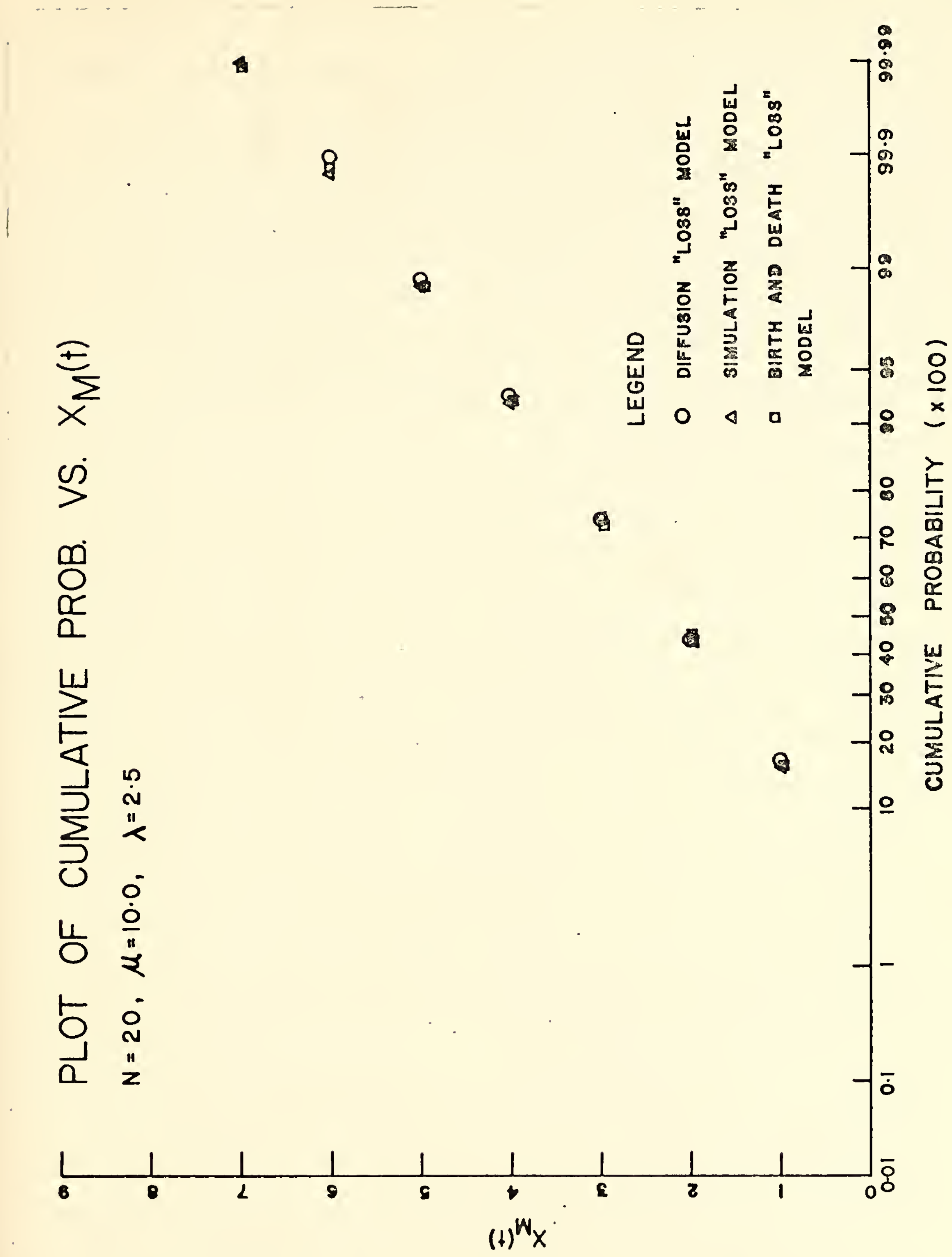

PLOT OF CUMULATIVE PROB. VS. $X_M(t)$
N = 20, μ = 10·0, λ = 2·5
LEGEND
○ DIFFUSION "LOSS" MODEL
△ SIMULATION "LOSS" MODEL
□ BIRTH AND DEATH "LOSS" MODEL
$X_M(t)$
0 1 2 3 4 5 6 7 8 9
0·01 0·1 1 10 20 30 40 50 60 70 80 90 95 99 99·9 99·99
CUMULATIVE PROBABILITY (x 100)

Figure 3

VIII. APPROXIMATION OF "NO LOSS" MODEL BY "LOSS" MODEL

Since the diffusion "loss" model gives relatively simple formulas for x_s, σ_s^2 and P_j's, it appeared desirable to relate the results of the "loss" model to the "no loss" model. It may be noted that in the "loss" model, only a fraction of the arriving messages are transmitted, whereas in the GPSS "no loss" model all arriving messages are eventually transmitted.

First, we need to determine the fraction of messages transmitted in order to obtain the effective rate of message arrivals at a "loss" model. In the "loss" model, the probability that a message arrives at a busy machine is approximately x_s divided by M or y_e, assuming steady state. Since a message can be transmitted only if both the transmitting and receiving machines are available, the probability of a message being transmitted immediately is approximately $(1-y_e)^2$ assuming independence between availability of machines and that N is large. Hence λ_{eff} is the effective arrival rate of messages at a machine in the "loss" model is given by

$$\lambda_{eff} = \lambda(1-y_e)^2 \tag{8.1}$$

This leads to the idea that a "no loss" model with input parameters N, λ and μ can be approximated by a "loss" model with input parameters N, λ' and μ where λ' is such that $\lambda'(1-y_e')^2 = \lambda$. Using equation (4.15) for y_e' with $\rho' = \frac{\mu}{2\lambda'}$,

the equation $\lambda'(1-y_e')^2 = \lambda$ can be rewritten as

$$\lambda'[\sqrt{(1+\frac{\mu}{4\lambda'})^2-1} - \frac{\mu}{4\lambda'}]^2 = \lambda \tag{8.2}$$

After more rearrangement of the various terms (details are given in Appendix E), we obtain an equation expressing λ' explicitly as a function of μ and λ:

$$\lambda' = \frac{\lambda}{(1-\frac{1}{\rho})^2} \tag{8.3}$$

The above relationship between "loss" and "no loss" models was tested using the simulation models with five different values of μ and λ' as input parameters, N and λ being kept constant at 20 and 2.5 respectively. The steady state means and variances are summarized in Table VI. For the case of $N=20$, $\mu=10.0$, $\lambda=2.5$ and $\lambda'=10.0$, the steady state cumulative probabilities are also calculated using the diffusion "loss" model, and the results from all three models are tabulated in Table VII and plotted in Figure 4. All the results indicate that the approximation of the "no loss" model by the "loss" model is quite accurate.

Table VI

Comparison of "no loss" model with its "equivalent" "loss" model, N=20, λ=2.5.

Simulation "no loss" model				Simulation "loss model			
λ	μ	x_s	σ_s^2	λ'	λ_{eff}	x_s	σ_s^2
2.50	7.50	6.677	1.362	22.500	2.50	6.694	1.343
2.50	8.57	5.758	1.503	14.407	2.50	5.862	1.556
2.50	10.00	4.972	1.640	10.000	2.50	5.031	1.659
2.50	12.00	4.105	1.732	7.347	2.50	4.192	1.726
2.50	15.00	3.319	1.651	5.625	2.50	3.357	1.680

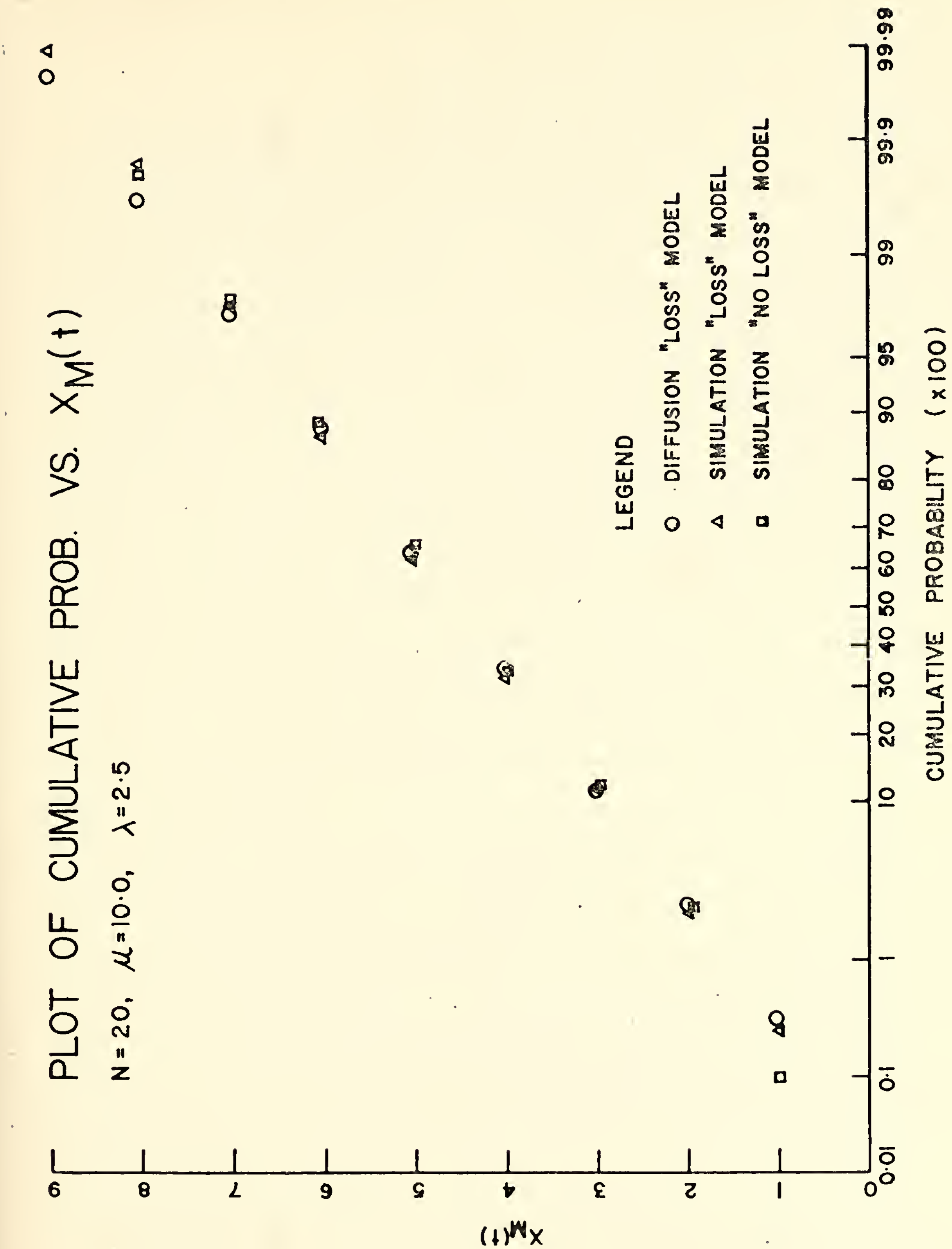
(Each simulation runs for 2400 hours)

Table VII

Steady State Cumulative Probabilities Calculated from Three Different Models, N=20, μ=10.0.

State j	Steady State Cumulative Probabilities P_j		
	Simulation "no loss" model λ=2.5	Simulation "loss" model λ'=10.0	Diffusion "loss" model λ'=10.0
0	0.0000	0.0001	0.0002
1	0.0010	0.0028	0.0034
2	0.0240	0.0245	0.0266
3	0.1210	0.1159	0.1146
4	0.3480	0.3347	0.3492
5	0.659	0.6380	0.6507
6	0.8900	0.8780	0.8874
7	0.9810	0.9790	0.9734
8	0.9980	0.9985	0.9966
9	1.0000	0.9999	0.9998
10	1.0000	1.0000	1.0000

(Each simulation runs for 2400 hours)



PLOT OF CUMULATIVE PROB. VS. $X_M(t)$
N = 20, μ = 10·0, λ = 2·5
LEGEND
○ DIFFUSION "LOSS" MODEL
△ SIMULATION "LOSS" MODEL
□ SIMULATION "NO LOSS" MODEL
$X_M(t)$
CUMULATIVE PROBABILITY (x100)
0
1
2
3
4
5
6
7
8
9
0·01
0·1
1
10
20
30
40
50
60
70
80
90
95
99
99·9
99·99

Figure 4

IX. CONCLUSION

It was found that the steady state results of the diffusion and the birth and death models were in very close agreement with the simulation model, through comparison of the steady state means, variances and cumulative probabilities. The usefulness of the diffusion model, as compared to the birth and death model, lies in the ease with which its steady state mean and variance and its discretized probabilities can be computed without resorting to a digital computer.

The simple form of equations (7.1) and (7.2) exhibits the fact that the steady state mean and variance vary linearly with N and depend only on ρ, and not on the individual values of λ and μ. These results cannot be inferred from the steady state formulas of the birth and death model or the simulation model.

Lastly, we developed a simulation model for the much more complicated case when the messages are allowed to wait for transmission. We showed that a "no loss" model with input parameter N, λ and μ can be quite accurately approximated by a "loss" model with input parameter N, λ', and μ where λ' is calculated through equation (8.3) from the other parameters.

APPENDIX A

Derivation of the Characteristic Function of S(t)

Define $\phi_M(\theta,t)$ and $\psi_M(\theta,t)$ as the characteristic functions of $X_M(t)$ and $S_M(t)$ respectively:

$$\phi_M(\theta,t) = E[e^{i\theta X_M(t)}] \tag{A.1}$$

$$\psi_M(\theta,t) = E[e^{i\theta S_M(t)}] \tag{A.2}$$

Since

$$X_M(t) = My(t) + \sqrt{M}\, S_M(t)$$

we obtain

$$\psi_M(\theta,t) = e^{-\theta\sqrt{M}y(t)}\ \phi_M(\theta/\sqrt{M},\ t) \tag{A.3}$$

Given that $X_M(t) = j$, equations (4.3), (4.4) and (4.5) give the respective probabilities that $X_M(t+dt)$ be equal to j+1, j-1 or j. Hence we can write down the following expression for $\phi_M(\theta,t+dt)$:

$$\begin{aligned}\phi_M(\theta,t+dt) &= E[e^{i\theta X_M(t+dt)}]\\ &= E[e^{i\theta(X_M(t)+1)}\{4M^2-2M+2X_M(t)(1-4M)+4X_M(t)\}\frac{\lambda dt}{2M-1}\\ &+ e^{i\theta X_M(t)}\{1-2X_M(t)\rho\lambda dt-(4M^2-2M+2X_M(t)(1-4M)\\ &+4X_M^2(t))\frac{\lambda dt}{2M-1}\} + e^{i\theta(X_M(t)-1)}\{2X_M(t)\rho\lambda dt\} + o(dt)]\end{aligned} \tag{A.4}$$

Then after rearranging the terms and taking the limit as $t \to \infty$, we obtain

$$\frac{\partial\phi_M(\theta,t)}{\partial t} = \lim_{dt\to 0} \frac{\phi_M(\theta,t+dt)-\phi_M(\theta,t)}{dt}$$

$$= E[e^{i\theta X_M(t)}e^{i\theta}\{4M^2-2M+2X_M(t)(1-4M)+4X_M^2(t)\}\frac{\lambda}{2M-1}$$

$$+ e^{i\theta X_M(t)}\{-2\rho\lambda X_M(t)-(4M^2-2M+2X_M(t)(1-4M)+4X_M^2(t))$$

$$\frac{\lambda}{2M-1}\} + e^{i\theta X_M(t)}e^{-i\theta}\{2\rho\lambda X_M(t)\}] \tag{A.5}$$

Further rearranging of terms results in

$$\frac{\partial\phi_M(\theta,t)}{\partial t} = E[X_M^2(t)e^{i\theta X_M(t)}(e^{i\theta}-1)\frac{4\lambda}{2M-1}$$

$$+ 2\lambda X_M(t)e^{i\theta X_M(t)}\{\frac{1-4M}{2M-1}(e^{i\theta}-1)+\rho(e^{-i\theta}-1)\}$$

$$+ 2\lambda Me^{i\theta X_M(t)}(e^{i\theta}-1)] \tag{A.6}$$

However, since

$$\frac{\partial\phi_M(\theta,t)}{\partial\theta} = E[iX_M(t)e^{i\theta X_M(t)}]$$

$$\frac{\partial^2\phi_M(\theta,t)}{\partial\theta^2} = E[-X_M^2(t)e^{i\theta X_M(t)}]$$

the above can now be written as

$$\frac{\partial\phi_M(\theta,t)}{\partial t} = \frac{\partial^2\phi_M(\theta,t)}{\partial\theta^2}(1-e^{i\theta})\frac{4\lambda}{2M-1} + 2\lambda M\phi_M(\theta,t)(e^{i\theta}-1)$$

$$+ 2\lambda i\frac{\partial\phi_M(\theta,t)}{\partial\theta}\{\frac{4M-1}{2M-1}(e^{i\theta}-1)-\rho(e^{-i\theta}-1)\} \tag{A.7}$$

Rewriting equation (A.3) we obtain the following expressions for $\phi_M(\theta/\sqrt{M},t)$ and its first and second partial derivatives

$$\phi_M(\theta/\sqrt{M},t) = e^{i\theta\sqrt{M}y(t)}\psi_M(\theta,t) \tag{A.8}$$

$$\frac{\partial\phi_M(\theta/\sqrt{M},t)}{\partial t} = e^{i\theta\sqrt{M}y(t)}\{\frac{\partial\psi_M(\theta,t)}{\partial t} + i\theta\sqrt{M}y'(t)\psi_M(\theta,t)\} \tag{A.9}$$

$$\frac{\partial\phi_M(\theta/\sqrt{M},t)}{\partial\theta} = e^{i\theta\sqrt{M}y(t)}\{\frac{\partial\psi_M(\theta,t)}{\partial\theta} + i\sqrt{M}y(t)\psi_M(\theta,t)\} \tag{A.10}$$

$$\frac{\partial^2\phi_M(\theta/\sqrt{M},t)}{\partial\theta^2} = e^{i\theta\sqrt{M}y(t)}\{\frac{\partial^2\psi_M(\theta,t)}{\partial\theta^2} + 2i\sqrt{M}y(t)\frac{\partial\psi_M(\theta,t)}{\partial\theta}$$

$$-My^2(t)\psi_M(\theta,t)\} \tag{A.11}$$

If we rewrite equation (A.7) with $\theta/\sqrt{M}$ in place of θ, we obtain

$$\frac{\partial\phi_M(\theta/\sqrt{M},t)}{\partial t} = \frac{\partial^2\phi_M(\theta/\sqrt{M},t)}{\partial\theta^2}(1-e^{i\theta/\sqrt{M}})\frac{4\lambda M}{2M-1}$$

$$+ 2\lambda\sqrt{M}i\frac{\partial\phi_M(\theta/\sqrt{M},t)}{\partial\theta}\{\frac{4M-1}{2M-1}(e^{i\theta/\sqrt{M}}-1)-\rho(e^{-i\theta/\sqrt{M}}-1)\}$$

$$+ 2\lambda M\phi_M(\theta/\sqrt{M},t)(e^{i\theta/\sqrt{M}}-1) \tag{A.12}$$

Using equations (A.8) through (A.11) to replace all the terms involving ϕ by ψ, and factoring out $e^{i\theta\sqrt{M}y(t)}$, equation (A.12) becomes

$$\frac{\partial\psi_M(\theta,t)}{\partial t} + i\theta\sqrt{M}y'(t)\psi_M(\theta,t)$$

$$= \frac{4\lambda M}{2M-1}(1-e^{i\theta/\sqrt{M}})\{\frac{\partial^2\psi_M(\theta,t)}{\partial\theta^2} + 2i\sqrt{M}y(t)\frac{\partial\psi_M(\theta,t)}{\partial\theta} -My^2(t)\psi_M(\theta,t)\}$$

$$+ 2\lambda\sqrt{M}i\{\frac{4M-1}{2M-1}(e^{i\theta/\sqrt{M}}-1)+\rho(1-e^{-i\theta/\sqrt{M}})\}\{\frac{\partial\psi_M(\theta,t)}{\partial\theta}$$

$$+i\sqrt{M}y(t)\psi_M(\theta,t)\} + 2\lambda M(e^{i\theta/\sqrt{M}}-1)\psi_M(\theta,t) \tag{A.13}$$

Substituting $e^{i\theta/\sqrt{M}}$ and $e^{-i\theta/\sqrt{M}}$ with their power series

$$1 - e^{i\theta/\sqrt{M}} = \frac{-i\theta}{\sqrt{M}} + \frac{\theta^2}{2M} + O(\frac{1}{M^{3/2}})$$

$$1 - e^{-i\theta/\sqrt{M}} = \frac{i\theta}{\sqrt{M}} + \frac{\theta^2}{2M} + O(\frac{1}{M^{3/2}})$$

and after more rearranging of terms in (A.13) we may write

$$\frac{\partial\psi_M}{\partial t} + i\theta\sqrt{M}\psi_M\{y'(t)-2\lambda(\frac{2M}{2M-1}y^2(t)-(\frac{4M-1}{2M-1} +\rho)y(t)-1)\}$$

$$= \frac{4\lambda M}{2M-1}(\frac{-i\theta}{\sqrt{M}} + \frac{\theta^2}{2M})\frac{\partial\psi_M{}^2}{\partial\theta^2} + \{\frac{4M}{2M-1}y(t)(\theta+ \frac{i\theta^2}{2\sqrt{M}})$$

$$- (\frac{4M-1}{2M-1})(\theta+ \frac{i\theta^2}{\sqrt{M}})+\rho(-\theta+ \frac{i\theta^2}{2\sqrt{M}})\}2\lambda\frac{\partial\psi_M}{\partial\theta} - \{\frac{2M}{2M-1}y^2(t)$$

$$- (\frac{4M-1}{2M-1} -\rho)y(t)+1\}2\lambda(\frac{\theta^2}{2})\psi_M+o(\frac{1}{\sqrt{M}}) \tag{A.14}$$

where the arguments of $\psi_M(\theta,t)$ have been dropped for ease of exposition. The deterministic process y(t) must satisfy the following first order differential equation, as we let $M \to \infty$

$$y'(t) -2\lambda \{y^2(t) - (2+\rho)y(t) -1\} = 0 \tag{A.15}$$

in order that equation (A.14) simplify to

$$\frac{\partial\psi}{\partial t} = -\theta\frac{\partial\psi}{\partial\theta}\{2\lambda(2+\rho-2y(t)\}- \frac{\theta^2}{2}\psi\ \{2\lambda(y^2(t)-(2-\rho)y(t)-1)\} \tag{A.16}$$

This is the transformed version of the forward differential equation for a non-stationary O.U. process.

We have to solve the first order differential equation (A.15) in order to obtain $y(t)$ as explicit function of t. For the sake of convenience, $y'(t)$ and $y(t)$ are written as $\frac{dy}{dt}$ and y respectively. Then equation (A.15) is

$$\frac{dy}{dt} = 2\lambda\{y^2-y(2+\rho)+1\} \tag{A.17}$$

Now let

$$v = 1 + \tfrac{1}{2}\rho$$

and

$$w^2 = (1+\tfrac{1}{2}\rho)^2 - 1$$

Then equation (A.17) can be written as

$$\frac{dy}{dt} = 2\lambda\{(y-v)^2-w^2\}$$

which in turn may be written as

$$\left(\frac{1}{y-v-w} - \frac{1}{y-v+w}\right)dy = 4\lambda w dt$$

Integrating both sides of the equation we find

$$\ln\frac{y-v-w}{y-v+w} = 4\lambda wt + c_1$$

or

$$\frac{y-v+w}{y-v-w} = c_2 e^{-4\lambda wt}$$

Assume that the system starts off empty, hence $y=0$ when $t=0$, then c_2 is evaluated as

$$c_2 = \frac{w-v}{w+v}$$

Now let

$$k = \frac{w-v}{w+v} e^{-4\lambda wt} = \frac{y-v+w}{y-v-w}$$

Then

$$y = v - \frac{1+k}{1-k} w$$

Rewriting y as a function of t again

$$y(t) = v - w \left\{\frac{1 + \frac{w-v}{w+v}e^{-4\lambda wt}}{1 - \frac{w-v}{w+v}e^{-4\lambda wt}}\right\} \tag{A.18}$$

Further,

$$y_e = \lim_{t\to\infty} y(t) = v-w \tag{A.19}$$

APPENDIX B

Derivation of Stochastic Differential Equation

From equations (4.21), (4.22) and (4.23)

$$dX_M(t) = \beta_m(x,t)dt + Z(t)\sqrt{\alpha_M(x,t)dt} \tag{B.1}$$

$$\beta_M(x,t) = \frac{\{2M-2X_M(t)\}\{2M-2X_M(t)-1\}}{2M-1}\lambda - 2X_M(t)\rho\lambda \tag{B.2}$$

$$\alpha_M(x,t) = \frac{\{2M-2X_M(t)\}\{2M-2X_M(t)-1\}}{2M-1}\lambda + 2X_M(t)\rho\lambda \tag{B.3}$$

where $Z(t)$ is a purely random Gaussian process with zero mean and unit variance. Using equation (4.1) we can express $X_M(t)$ in terms of $S_M(t)$

$$X_M(t) = \sqrt{M}S_M(t) + My(t)$$

therefore

$$dX_M(t) = \sqrt{M}dS_M(t) + My'(t)dt \tag{B.4}$$

Using equations (B.2), (B.3) and (B.4) to substitute for $\beta(x,t)$, $\alpha(x,t)$ and $dX_M(t)$ in equation (B.1) we obtain

$$\sqrt{M}dS_M(t) + My'(t)dt$$

$$= [\frac{\{2M-2\sqrt{M}S_M(t)-2My(t)\}\{2M-2\sqrt{M}S_M(t)-2My(t)-1\}}{2M-1}\lambda dt$$

$$- 2\rho\lambda\{\sqrt{M}S_M(t)+My(t)\}dt] + Z(t)[2\rho\lambda\{\sqrt{M}S_M(t)+My(t)\}dt$$

$$+ \frac{\{2M-2\sqrt{M}S_M(t)-2My(t)\}\{2M-2\sqrt{M}S_M(t)-2My(t)-1\}}{2M-1}\lambda dt]^{\frac{1}{2}}$$

Carrying out all the necessary multiplication, rearranging some of the terms and (since we are eventually going to let M go to infinity) approximating 2M-1 by 2M, we obtain the following expression that is exact in the limit.

$$\sqrt{M}dS_M(t) + My'(t)dt$$
$$\simeq \lambda dt[2M-4My(t)+2My^2(t)+S_M(t)\{4\sqrt{M}(y(t)-1)+\frac{1}{\sqrt{M}}\}$$
$$+2S_M{}^2(t)-2\rho\{\sqrt{M}S_M(t)+My(t)\}] + Z(t)\sqrt{\lambda dt}[2M-4My(t)+2My^2(t)$$
$$+S_M(t)\{4\sqrt{M}(y(t)-1)+\frac{1}{\sqrt{M}}\}+2S_M{}^2(t)+2\rho\{\sqrt{M}S_M(t)+My(t)\}]^{\frac{1}{2}}$$

Dividing by $\sqrt{M}$ throughout the whole equation and after more rearranging of terms, we have

$$dS_M(t) + \sqrt{M}y'(t)dt$$
$$\simeq 2\sqrt{M}\lambda dt\{y^2(t)-(2+\rho)y(t)+1\}+2\lambda S_M(t)dt\{2y(t)-2-\rho+\frac{1}{2M}\}$$
$$+\frac{2\lambda}{\sqrt{M}}S_M{}^2(t) + Z(t)[2\lambda dt\{y^2(t)-(2-\rho)y(t)+1\}$$
$$+\frac{2\lambda S_M(t)dt}{\sqrt{M}}\{2y(t)-2+\rho+\frac{1}{2M}\} + \frac{2\lambda}{M}S_M{}^2(t)]^{\frac{1}{2}} \qquad (B.5)$$

Now we let M go to infinity and assume that $y'(t) = 2\lambda\{y^2(t)-(2+\rho)y(t)+1\}$. Then with $S(t) = \lim_{M\to\infty} S_M(t)$ we have

$$dS(t) = 2\lambda\{2y(t)-2-\rho\}S(t)dt+Z(t)[2\lambda\{y^2(t)-(2-\rho)y(t)+1\}dt]^{\frac{1}{2}}$$

Again we have shown that the process S(t) is an Ornstein-Uhlenbeck process with the following infinitesimal mean and

variance, conditioned on $S(t)=s$.

$$\beta(s,t) = -\beta(t)s = -2\lambda\{2y(t)-2-\rho\}s$$

$$\alpha(s,t) = \alpha(t) = 2\lambda\{y^2(t)-y(t)(2-\rho)+1\}$$

APPENDIX C

Derivation of Steady State Mean and Variance of $X_M(t)$

From equation (4.6) we have

$$\frac{\partial\psi}{\partial t} = -\beta(t)\theta\frac{\partial\psi}{\partial\theta} - \alpha(t)\left(\frac{\theta^2}{2}\right)\psi \tag{C.1}$$

where

$$\beta(t) = 2\lambda\{2+\rho-2y(t)\}$$

$$\alpha(t) = 2\lambda\{y^2(t)-y(t)(2-\rho)+1\}$$

Let the mean and variance of S(t) be denoted by $\mu(t)$ and $\tau(t)$, and since S(t) is normally distributed, we can write its characteristic function $\psi(\theta,t)$ in terms of $\mu(t)$ and $\tau(t)$.

$$\psi(\theta,t) = e^{\{i\theta\mu(t)-\frac{1}{2}\theta^2\tau(t)\}} \tag{C.2}$$

Differentiating $\psi(\theta,t)$ with respect to θ and t then dropping the arguments of $\psi(\theta,t)$, we obtain

$$\frac{\partial\psi}{\partial t} = \{i\theta\mu'(t)-\tfrac{1}{2}\theta^2\tau'(t)\}e^{\{i\theta\mu(t)-\frac{1}{2}\theta^2\tau(t)\}} \tag{C.3}$$

$$\frac{\partial\psi}{\partial\theta} = \{i\mu(t)-\theta\tau(t)\}e^{\{i\theta\mu(t)-\frac{1}{2}\theta^2\tau(t)\}} \tag{C.4}$$

We substitute these equations for ψ, $\frac{\partial\psi}{\partial\theta}$, $\frac{\partial\psi}{\partial t}$ into equation (C.1). Cancelling out the factor $e^{\{i\theta\mu(t)-\frac{1}{2}\theta^2\tau(t)\}}$ we obtain

$$i\theta\mu'(t)-\tfrac{1}{2}\theta^2\tau'(t) = -\beta(t)\theta\{i\mu(t)-\theta\tau(t)\}-\tfrac{1}{2}\theta^2\alpha(t) \tag{C.5}$$

By equating coefficients of $i\theta$ and θ^2 on both sides, we obtain the following two first order differential equations:

$$\mu'(t) = -\beta(t)\mu(t) \tag{C.6}$$

$$\tau'(t) = -2\beta(t)\tau(t) + \alpha(t) \tag{C.7}$$

Equation (C.6) can be easily solved by separation of variables [Ref. 7], yielding

$$\mu(t) = \mu(0)e^{-\int_0^t \beta(x)dx} \tag{C.8}$$

It is reasonable to assume that the noise process S(t) starts off at time zero with zero mean and variance, hence $\mu(0)=0$ and $\tau(0)=0$.

Substituting the initial condition $\mu(0)=0$ into equation (C.8), we obtain

$$\mu(t) = 0 \qquad \text{for all } t \geq 0 \tag{C.9}$$

Equation (C.7) can be solved [Ref. 7] by the use of an integrating factor, giving us the following expression for $\tau(t)$, with $\tau(0)=0$

$$\tau(t) = e^{-2\int_0^t \beta(x)dx} \left\{\int_0^t \alpha(z)e^{2\int_0^z \beta(x)dx} dz\right\} \tag{C.10}$$

Note that $\beta(t)$ and $\alpha(t)$ are functions of $y(t)$ where $y(t)$ is a complex function of t, given by equation (A.18):

$$y(t) = v - w\left\{\frac{1 + \frac{w-v}{w+v}e^{-4\lambda wt}}{1 - \frac{w-v}{w+v}e^{-4\lambda wt}}\right\} \tag{C.11}$$

Therefore the two integrals on the right hand side of equation (C.10) are quite intractable, unless t is very large, in which case we are able to derive an approximation for $\tau(t)$.

For the typical values of λ and ρ that we are using, the term $e^{-4\lambda wt}$ in equation (C.11) vanishes very rapidly with t approaching infinity. For example, with $\lambda=2.5$ messages/hour, $\rho=2$ and $t=1.0$ hour, $e^{-4\lambda wt} = e^{-4(2.5)(\sqrt{3})(1)} = 3.0 \times 10^{-8}$. So for time t greater than one hour, $y(t)$ can be very accurately approximated by y_e where

$$y_e = 1+\tfrac{1}{2}\rho - \sqrt{(1+\tfrac{1}{2}\rho)^2-1}$$

and hence $\beta(t)$ and $\alpha(t)$ can be approximated by the following expressions.

$$\beta(t) \simeq 2\lambda(2+\rho-2y_e) = \beta_e \tag{C.12}$$

$$\alpha(t) \simeq 2\lambda(y_e{}^2-y_e(2-\rho)+1) = \alpha_e \tag{C.13}$$

Referring to Figure 5, we can see that when z is very large, such that the area under the curve $\beta(x)$ up to time $x=1.0$ hour, is insignificant when compared to the area under the curve up to time $x=z$, then $\beta_e z$ is a good approximation for $\int_0^z \beta(x)dx$.

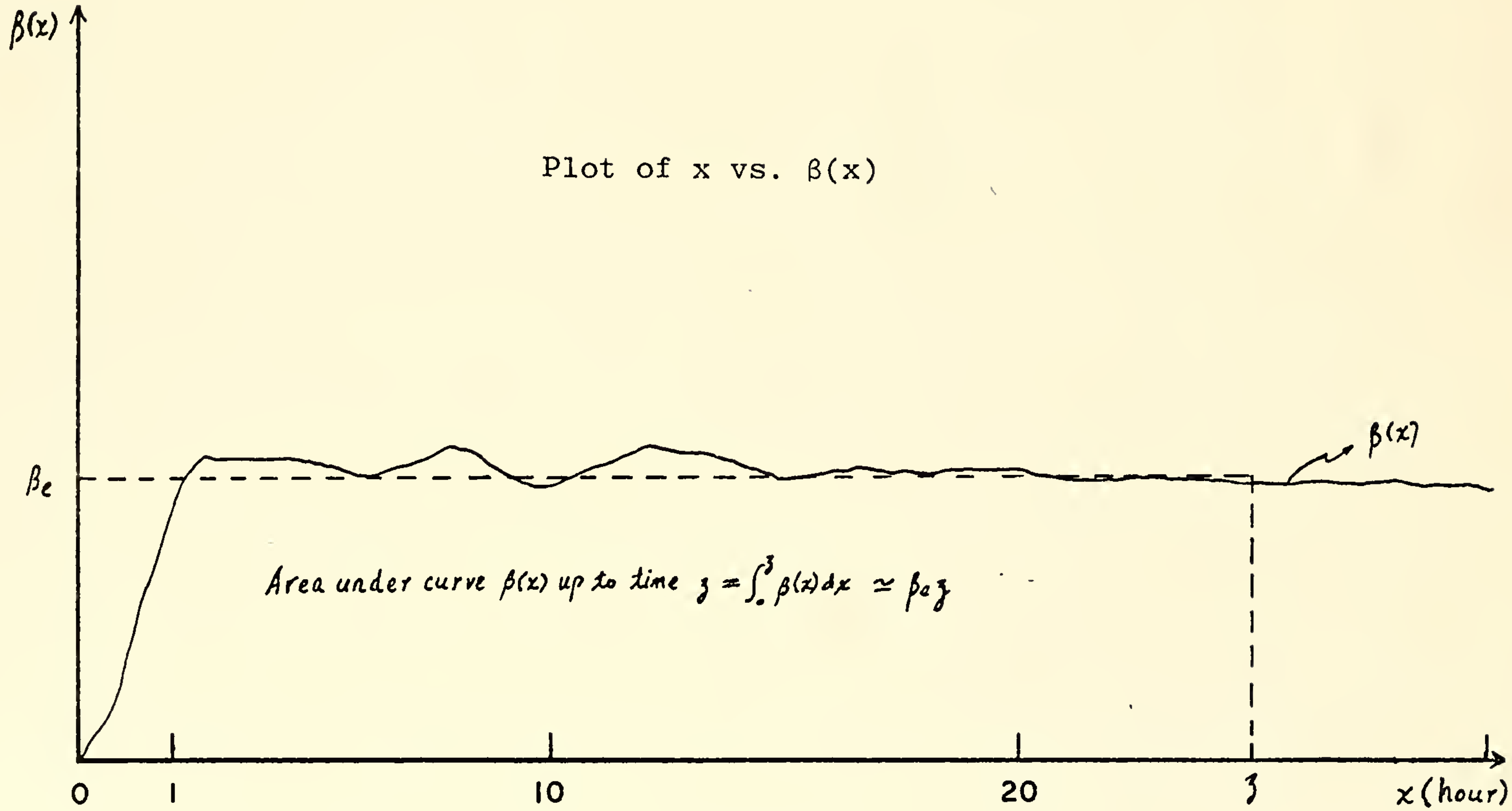


Figure 5

Not to scale

Similarly $\alpha(z)e^{2\int_0^z \beta(x)dx}$ can be approximated by $\alpha_e e^{2\beta_e z}$.

Thus an approximate solution for $\tau(t)$ is

$$\tau(t) \simeq \frac{\alpha_e}{2\beta_e}\left[1 - e^{-2\beta_e t}\right] \qquad t \geq 10 \text{ hours}$$

However, for such large value of t, $e^{-2\beta_e t} \simeq 0$, so we have

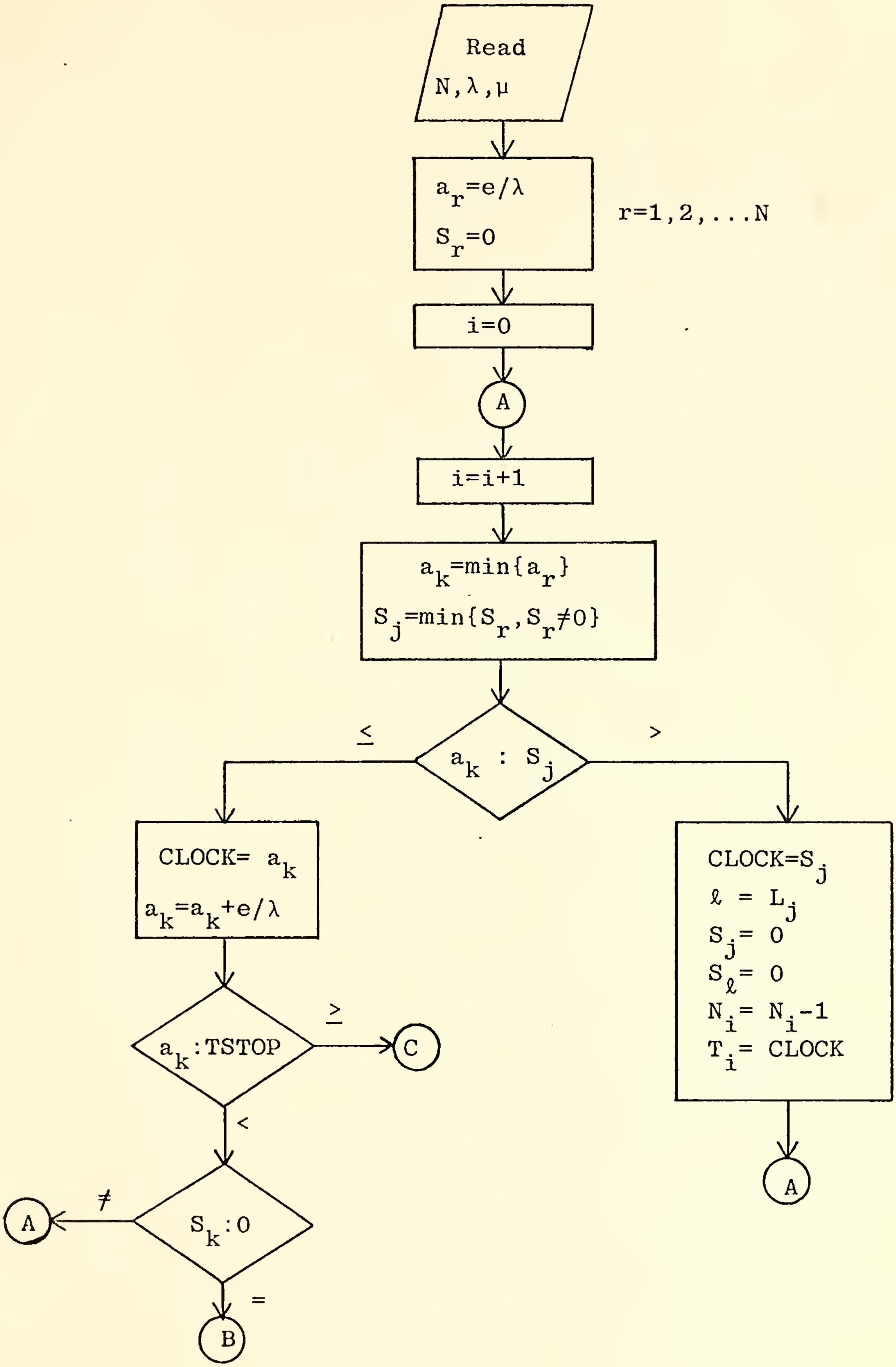
$$\tau(t) \simeq \frac{\alpha_e}{2\beta_e} \qquad t \geq 10 \text{ hours}$$

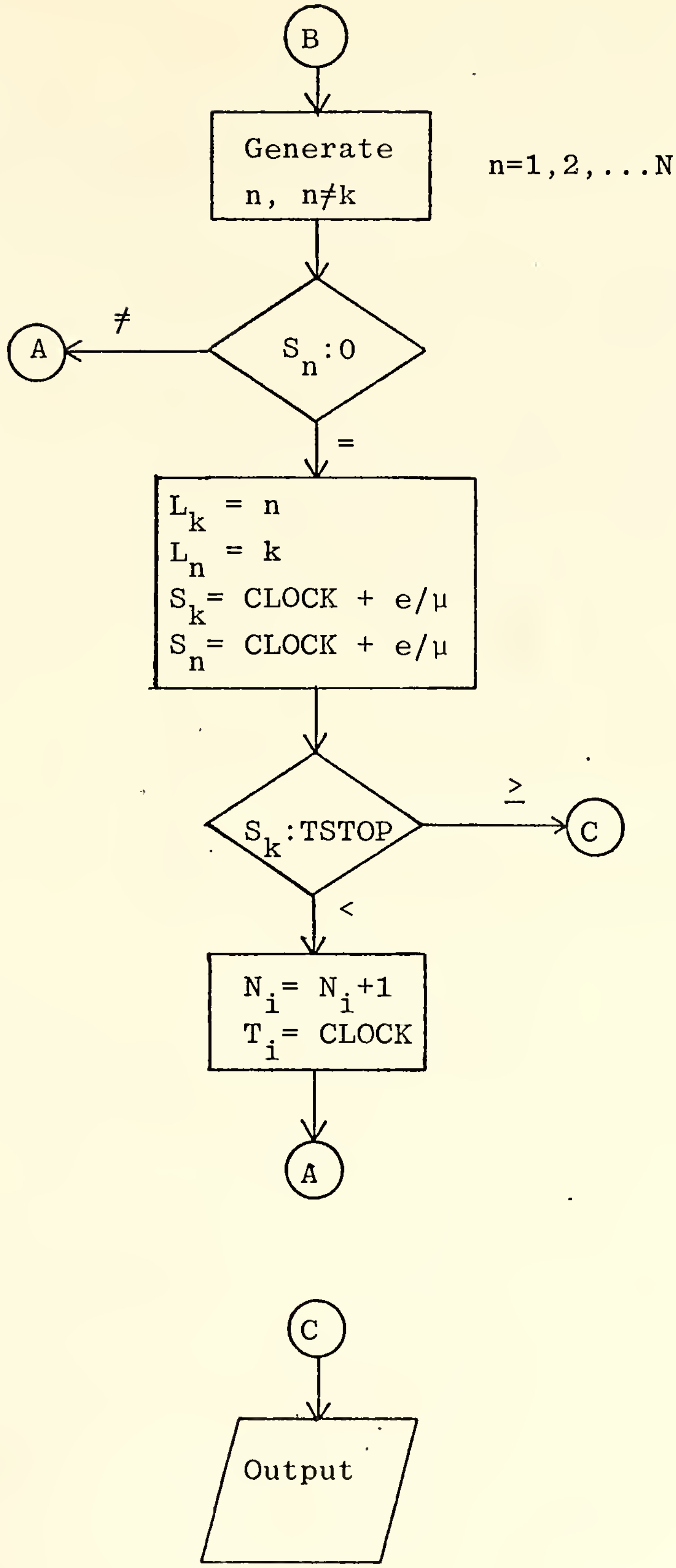
$$= \frac{y_e^2 - y_e(2-\rho)+1}{2(2+\rho-2y_e)}$$

APPENDIX D

Notations used in Simulation Program I

a_r	arrival time of next message at machine r
S_r	departure time of next message at machine r,
	S_r=0 when machine r is not in use
L_r	denotes the machine to which machine r is transmitting
N_i	number of messages undergoing transmission at epoch T_i
CLOCK	current clock time of simulation model
e	exponential variate with mean = 1
λ	arrival rate of messages at each machine
μ	departure rate of a message undergoing transmission
N	total number of machines
n	discrete uniform (0,N) variate
TSTOP	length of one simulation run


Read
N, λ, μ
$a_r = e/\lambda$
$S_r = 0$
r=1,2,...N
i=0
A
i=i+1
$a_k = \min\{a_r\}$
$S_j = \min\{S_r, S_r \neq 0\}$
≤
$a_k : S_j$
>
CLOCK= a_k
$a_k = a_k + e/\lambda$
CLOCK=S_j
$\ell = L_j$
$S_j = 0$
$S_\ell = 0$
$N_i = N_i - 1$
T_i = CLOCK
A
≥
a_k : TSTOP
C
<
≠
A
$S_k : 0$
=
B


B
Generate
n, n≠k
n=1,2,...N
≠
A
S_n:0
=
L_k = n
L_n = k
S_k= CLOCK + e/μ
S_n= CLOCK + e/μ
≥
C
S_k:TSTOP
<
N_i= N_i+1
T_i= CLOCK
A
C
Output

```
C
C
C     . SIMULATION 'LOSS' MODEL
C       PROGRAM I
C
C
C
C
C       SIMULATE A SYSTEM OF N SENDOX MACHINES
C       ASSUMING THAT ANY ARRIVAL THAT FAILS
C       TO GET ON THE SYSTEM IS LOST
C
C
C
C
C       A(J)             TIME OF NEXT ARRIVAL AT MACHINE J
C     . S(J)             TIME OF NEXT DEPARTURE AT MACHINE J
C                        = 0 IF MACHINE IS NOT IN USE
C       LINK(J)          CONTAIN THE NUMBER OF THE-MACHINE THAT
C                        IS RECEIVING A MESSAGE OR TRANSMITTING
C                        TO MACHINE J
C       NSYS             NUMBER OF BUSY MACHINES
C       N                TOTAL NUMBER OF MACHINES
C       M                MAXIMUM NUMBER OF MESSAGES THAT CAN
C                        BE TRANSMITTED , = N/2
C       TIME             RECORDS THE INSTANT OF ARRIVAL OR
C                        DEPARTURE
C       DELT             DISCRETE TIME INTERVAL AT WHICH
C                        RELATIVE FREQUENCY ARE COLLECTED
C       P4               TIME AVERAGE PROBABILITY
C       P5               DISCRETE TIME PROBABILITY
C       PDF4             CUMULATIVE PROBABILITY FOR P4
C       PDF5             CUMULATIVE PROBABILITY FOR P5
C       ARRM             RECIPROCAL OF ARRIVAL RATE OF MESSAGE AT
C                        EACH MACHINE IN UNITS OF MINUTES
C       DEPM             RECIPROCAL OF DEPARTURE RATE OF
C                        TRANSMITTED MESSAGE IN UNITS OF MINUTES
C       TSTOP            LENGTH OF EACH SIMULATION RUN IN MINUTES
C       NRUN             NUMBER OF SIMULATION RUNS
C       IX               RANDOM NUMBER SEED
C       CLOCK            CURRENT CLOCK TIME
C       NARR             NUMBER OF MESSAGES THAT ARRIVE
C       NCN              NUMBER OF MESSAGES THAT ARRIVE AND ARE
C                        TRANSMITTED
C
C
C
      DIMENSION A(200), S(200), LINK(200)
     2, NSYS(5000), TIME(5000)
     3, P1(101), P2(101), P3(101), Y(101)
     4, DELT(1000)
     5, P4(101), P5(101)
     6, PDF4(101), PDF5(101)
      CALL OVFLOW
C
C     INITIALIZATION
C
      READ (5,1000) NRUN
      WRITE(6,1000) NRUN
      READ  (5,1000) N
      WRITE (6,1000) N
 1000 FORMAT (I5)
      READ  (5,2000) ARRM,DEPM,TSTOP
      WRITE (6,2000) ARRM,DEPM,TSTOP
 2000 FORMAT (3F10.2)
      M1=N/2+1
      DO 10 I=1,M1
      P4(I)=0.0
      P5(I)=0.0
   10 CONTINUE
      IX=52841
C
C     START OF SIMULATION RUNS
C
```

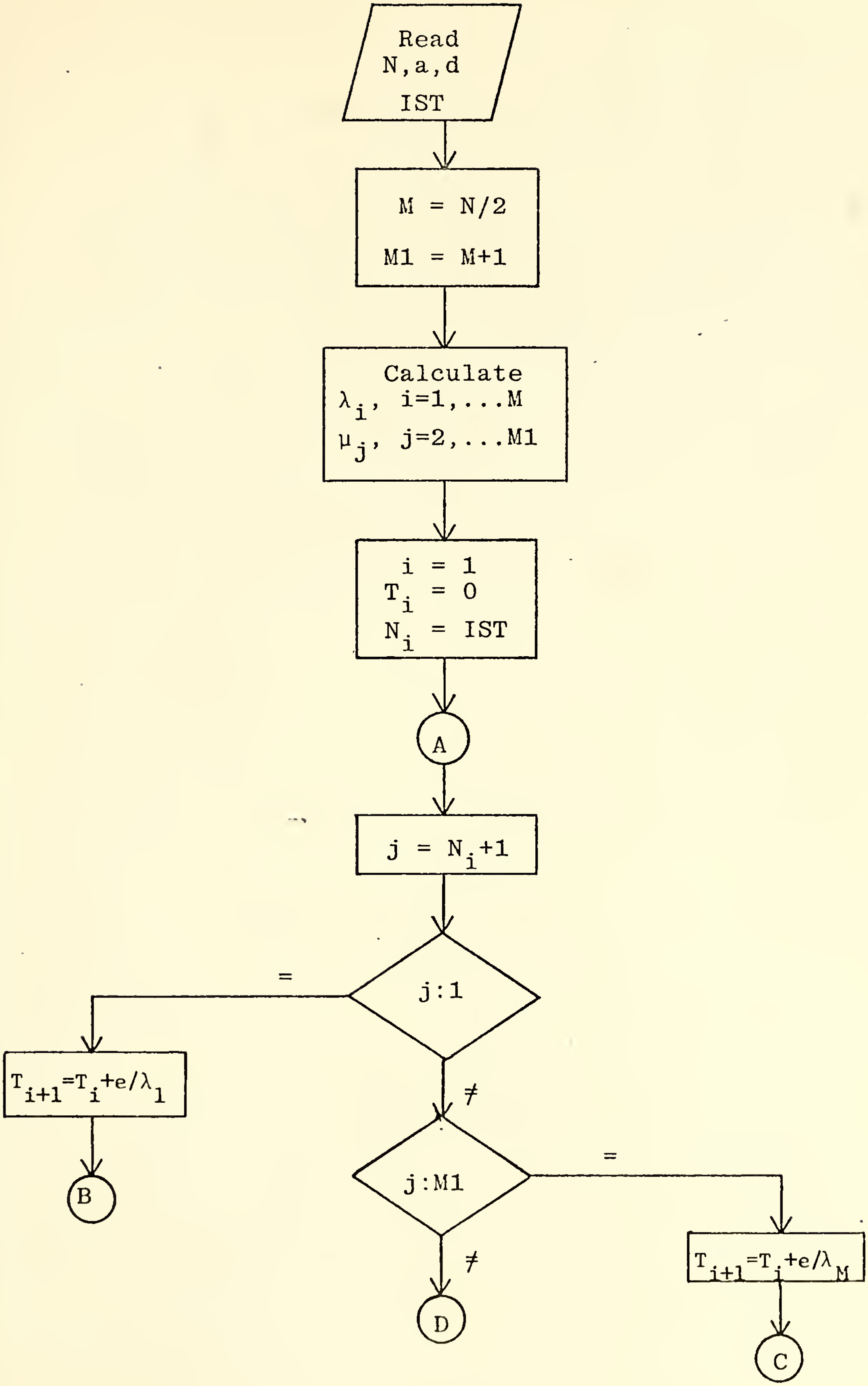
```
      DO 95 IC=1,NRUN
      IX=IX+IC*246
      CLOCK=0.0
      NARR=0
      NON=0
      JAD=1
      NSYS(JAD)=0
      TIME(JAD)=CLOCK
C
C     AT THE START OF EACH SIMULATION RUN  SET ALL S(K)=0
C     AND SET A(K) EQUAL TO TIME OF FIRT ARRIVAL
C
C
      DO 20 I=1,N
      S(I)=0
      CALL EXPON (IX,E,1)
      A(I)=E*ARRM
   20 CONTINUE
      DO 25 I=1,M1
      P1(I)=0.0
      P2(I)=0.0
   25 CONTINUE
   30 CONTINUE
C
C     SUBROUTINE ARNEXT SEARCHES FOR MIN A(K),K=1,N
C     SUBROUTINE ARNEXT SEARCHES FOR MIN S(K),K=1,N,S(K)#0
C
      CALL ARNEXT (A,K,N)
      CALL DPNEXT (S,L,N,SM)
      IF (A(K) .GE. SM) GO TO 50
      IF (A(K) .GE. TSTOP) GO TO 60
C
C     THE NEXT EVENT IS AN ARRIVAL AT MACHINE K
C     TEST TO SEE WHETHER IT WILL BE TRANSMITTED
C
      CLOCK=A(K)
      CALL EXPON (IX,E,1)
      A(K)=A(K)+E*ARRM
      NARR=NARR+1
      IF (S(K) .GE. CLOCK) GO TO 30
   40 CONTINUE
C
C     GENERATE DESTINATION MACHINE NDES=1,N,NDES#K
C
      CALL RANDOM (IX,Z,1)
      NDES=Z*N
      NDES=NDES+1
      IF (NDES .EQ. K) GO TO 40
      IF (S(NDES) .GE. CLOCK) GO TO 30
      CALL EXPON (IX,E,1)
      E=E*DEPM
      S(NDES)=CLOCK+E
      LINK(NDES)=K
      S(K)=CLOCK+E
      LINK(K)=NDES
      JAD=JAD+1
      NON=NON+1
      NSYS(JAD)=NSYS(JAD-1)+1
      TIME(JAD)=CLOCK
      GO TO 30
   50 CONTINUE
C
C     THE NEXT EVENT IS A DEPARTURE AT MACHINE L
C     AND MACHINE JL (=LINK(L))
C
      IF (SM    .GE. TSTOP) GO TO 60
      JL=LINK(L)
      S(JL)=0
      CLOCK=S(L)
      S(L)=0
      JAD=JAD+1
      NSYS(JAD)=NSYS(JAD-1)-1
```

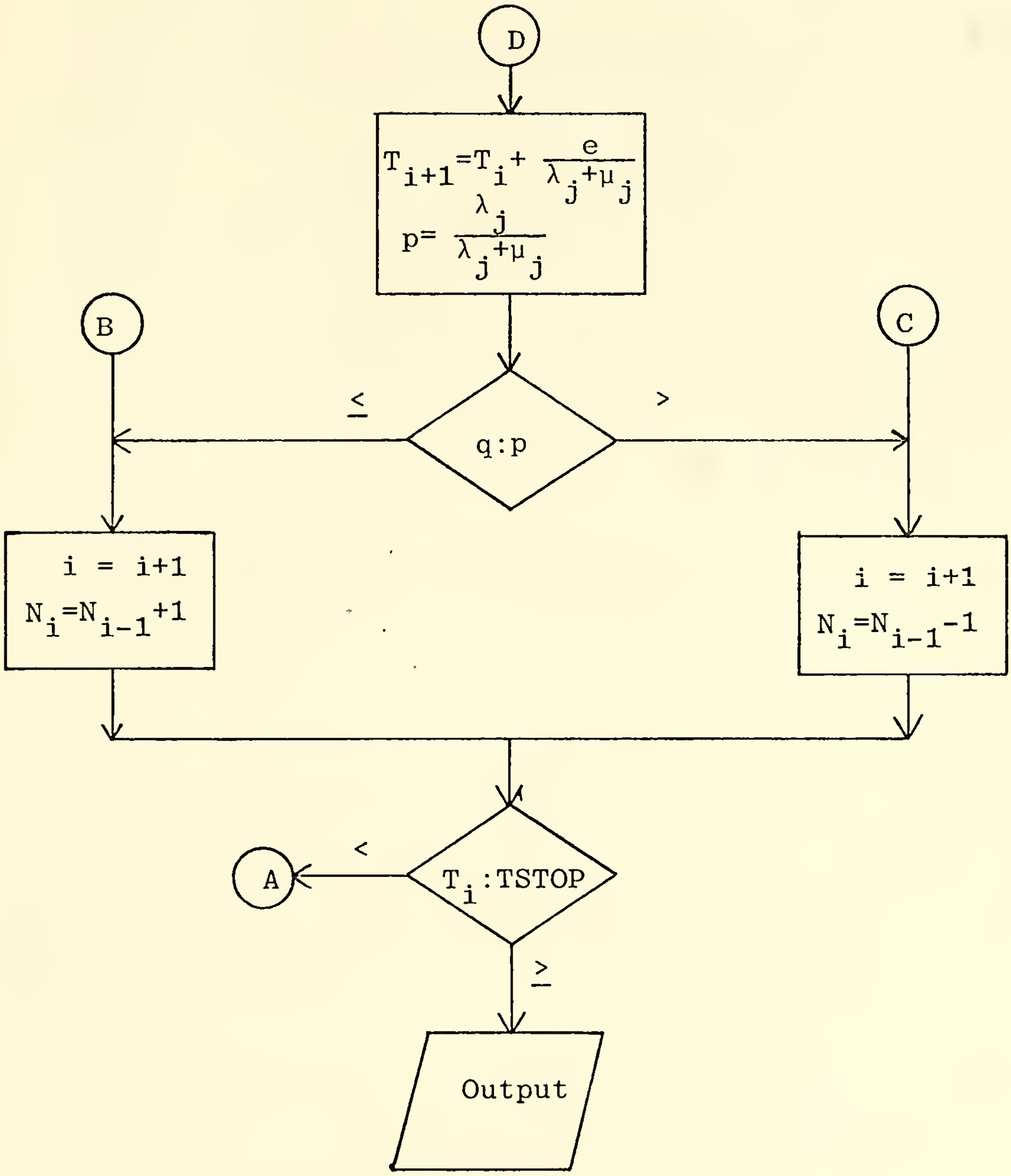
```
      TIME(JAD)=CLOCK
      GO TO 30
   60 CONTINUE
      TIME(JAD+1)=TSTOP
C
C     COMPUTE TIME AVERAGE PROBABILITIES
C
      DO 70 I=1,JAD
      II=NSYS(I)+1
      P1(II)=P1(II)+TIME(I+1)-TIME(I)
   70 CONTINUE
C
C     COMPUTE DISCRETE TIME PROBABILITIES
C
      JJ=1
      NSTOP=TSTOP+0.1
      DO 80 I=1,NSTOP
      DELT(I)=FLOAT(I)
   75 CONTINUE
      JJ=JJ+1
      IF (DELT(I) .GT. TIME(JJ)) GO TO 75
      JJ=JJ-1
      KK=NSYS(JJ)+1
      P2(KK)=P2(KK)+1.0
   80 CONTINUE
      DO 90 I=1,M1
      Y(I)=FLOAT(I-1)
      P1(I)=P1(I)/TSTOP
      P2(I)=P2(I)/TSTOP
      P4(I)=P4(I)+P1(I)
      P5(I)=P5(I)+P2(I)
   90 CONTINUE
      IX=52841
   95 CONTINUE
      SP4=0.0
      SP5=0.0
      VP4=0.0
      VP5=0.0
C
C     COMPUTE STEADY STATE MEAN, VARIANCE AND
C     CUMULATIVE PROBABILITIES USING P4 & P5
C
      DO 99 I=1,M1
      P4(I)=P4(I)/NRUN
      P5(I)=P5(I)/NRUN
      SP4=SP4+P4(I)*(I-1)
      SP5=SP5+P5(I)*(I-1)
      VP4=VP4+P4(I)*(I-1)*(I-1)
      VP5=VP5+P5(I)*(I-1)*(I-1)
      IF (I .NE. 1) GO TO 96
      PDF4(I)=P4(I)
      PDF5(I)=P5(I)
      GO TO 97
   96 CONTINUE
      PDF4(I)=PDF4(I-1)+P4(I)
      PDF5(I)=PDF5(I-1)+P5(I)
   97 CONTINUE
      II=I-1
      WRITE (6,8200) II, P3(I), P4(I), P5(I), PDF4(I)
     1, PDF5(I)
 8200 FORMAT (I10,5F12.4,/)
   99 CONTINUE
      WRITE (6,8800)
 8800 FORMAT (////)
      VP4=VP4-SP4**2
      VP5=VP5-SP5**2
C
C     PRINTOUT OF RESULTS
C
      WRITE(6,7000) NRUN,SP4,VP4
 7000 FORMAT (10X,'TIME AVERAGE PROBABILITY WITH',I5,
     1 ' SIMULATION RUNS',/,10X,'MEAN = ',F10.4,'VARIANCE ='
```

```
     2 F10.4,//)
       WRITE(6,8000) NRUN,SP5,VP5
8000   FORMAT (10X, 'DISCRETE TIME PROBABILITY WITH',I5,
     1 ' SIMULATION RUNS',/,10X,'MEAN = ',F10.4,
     2 '   VARIANCE =',F10.4,////)
       CALL PLOTP (Y,P3,M1,0)
       WRITE (6,8800)
       CALL PLOTP (Y,P4,M1,0)
       WRITE (6,8800)
       CALL PLOTP (Y,P5,M1,0)
       WRITE (6,8800)
       STOP
       END
       SUBROUTINE ARNEXT (A,K,N)
       DIMENSION A(100)
       K=1
       AM=100000.0
       DO 50 I=1,N
       IF (A(I) .GE. AM) GO TO 50
       AM=A(I)
       K=I
 50    CONTINUE
       RETURN
       END
       SUBROUTINE DPNEXT (S,L,N,SM)
       DIMENSION S(100)
       L=1
       SM=2000000.0
       DO 50 I=1,N
       IF (S(I) .LE. 0.00001) GO TO 50
       IF (S(I) .GE. SM) GO TO 50
       SM=S(I)
       L=I
 50    CONTINUE
       RETURN
       END
```

Notations used in Simulation Program II

a	arrival rate of messages at each machine
d	departure rate of a message undergoing transmission
N	total number of machines
M	= N/2
λ_i	rate of entering state i
μ_i	rate of leaving state i
IST	initial state at time 0
N_i	number of messages undergoing transmission at epoch T_i
e	exponential variate with mean = 1
q	uniform (0,1) variate
TSTOP	length of one simulation run


Read
N,a,d
IST
M = N/2
M1 = M+1
Calculate
λ_i, i=1,...M
μ_j, j=2,...M1
i = 1
T_i = 0
N_i = IST
A
$j = N_i+1$
j:1
=
≠
$T_{i+1}=T_i+e/\lambda_1$
B
j:M1
=
≠
$T_{i+1}=T_i+e/\lambda_M$
D
C


D
$T_{i+1}=T_i+\frac{e}{\lambda_j+\mu_j}$
$p=\frac{\lambda_j}{\lambda_j+\mu_j}$
B
C
≤
>
q:p
i = i+1
$N_i=N_{i-1}+1$
i = i+1
$N_i=N_{i-1}-1$
<
A
T_i:TSTOP
≥
Output

```
C
C
C     SIMULATION 'LOSS' MODEL
C     PROGRAM II
C
C
C     TO SIMULATE THE TRANSITION OF STATES IN A
C     BIRTH AND DEATH PROCESS
C
C
C     NRUN              NUMBER OF SIMULATION RUNS
C     M                 MAXIMUM NUMBER OF MESSAGES THAT
C                       CAN BE TRANSMITTED
C     RARR              RATE OF ARRIVAL OF MESSAGES AT EACH
C                       MACHINE
C     RDEP              RATE OF DEPARTURE OF A MESSAGE
C                       UNDERGOING TRANSMISSION
C     RL(J)             RATE OF ENTERING STATE J
C     RM(J)             RATE OF LEAVING STATEJ
C     IX                RANDOM NUMBER SEED
C     TSTOP             LENGTH OF EACH SIMULATION RUN IN MINUTES
C     NSYS(J)           NUMBER OF MESSAGES UNDERGOING
C                       TRANSMISSION AT EPOCH T(J)
C     IST               INITIAL STARTING STATE
C     P1, P4            STEADY STATE TIME AVERAGE PROBABILITIES
C                       FOR 1, 200 SIMULATION RUNS RESPECTIVELY
C     P2, P5            STEADY STATE DISCRETE TIME PROBABILITIES
C                       FOR 1, 200 SIMULATION RUNS RESPECTIVELY
C     PDF4              CUMULATIVE STEADY STATE PROBABILITIES P4
C     PDF5              CUMULATIVE STEADY STATE PROBABILITIES P5
C
      DIMENSION RL(101), RM(101)
     1, NSYS(5000), T(5000)
     2, P1(101), P2(101), P3(101), X(101)
     3, DELT(1000)
     4, PDF4(101), PDF5(101)
     5, P4(101), P5(101)
      CALL OVFLOW
C
C     INITIALIZATION
C
      READ (5,1000) NRUN
      WRITE(6,1000) NRUN
      READ (5,1000) M
      WRITE(6,1000) M
      READ (5,1000) IST
      WRITE(6,1000) IST
 1000 FORMAT (I5)
      READ (5,2000) RARR, RDEP, TSTOP
      WRITE(6,2000) RARR, RDEP, TSTOP
 2000 FORMAT (3F10.4)
      RARR=1.0/RARR
      RDEP=1.0/RDEP
      M1=M+1
      M2=2*M
      RARR=RARR/(M2-1)
      P4(1)=0.0
      P5(1)=0.0
C
C     COMPUTE RATES OF ENTERING AND
C     LEAVING THE VARIOUS STATES
C
      DO 10 I=2,M1
      I2=2*(I-2)
      RL(I-1)=(M2-I2)*RARR*(M2-I2-1)
      RM(I)=(I-1)*RDEP
      P4(I)=0.0
      P5(I)=0.0
   10 CONTINUE
      IX=35641
C
```

```
C       START OF SIMULATION RUNS
C       REPEATED NRUN TIMES
C
        DO 95 NR=1,NRUN
        IX=IX+NR*513
        DO 15 I=1,M1
        P1(I)=0.0
        P2(I)=0.0
   15   CONTINUE
C
C       AT THE BEGINNING OF EACH RUN SET TIME T(0)=0
C       AND NSYS(0)=IST
C
        IND=1
        T(IND)=0.0
        NSYS(IND)=IST
   20   CONTINUE
        CALL EXPON (IX,E,1)
        J=NSYS(IND)+1
        IF (J .EQ. 1) GO TO 30
        IF (J .EQ. M1) GO TO 40
        RATE=RL(J)+RM(J)
C
C       T(IND+1) DENOTES THE INSTANT WHEN THE PROCESS
C       CHANGES STATE
C
        T(IND+1)=T(IND)+E/RATE
        IF (T(IND+1) .GE. TSTOP) GO TO 70
C
C       TO DETERMINE WHETHER PROCESS GOES TO STATE
C       J+1 OR J-1
C
        PB=RL(J)/RATE
        CALL RANDOM (IX,Y,1)
        IF (Y .LE. PB) GO TO 50
        GO TO 60
   30   CONTINUE
C
C       PROCESS IS IN STATE 0 ALWAYS GOES TO STATE 1
C
        T(IND+1)=T(IND)+E/RL(J)
        IF (T(IND+1) .GE. TSTOP) GO TO 70
        GO TO 50
   40   CONTINUE
C
C       PROCESS IS IN STATE M ALWAYS GOES TO STATE M-1
C
        T(IND+1)=T(IND)+E/RM(J)
        IF (T(IND+1) .GE. TSTOP) GO TO 70
        GO TO 60
   50   CONTINUE
C
C       PROCESS GOES TO STATE J+1
C
        K=NSYS(IND)+1
        IND=IND+1
        NSYS(IND)=K
        P1(K)=P1(K)+T(IND)-T(IND-1)
        GO TO 20
   60   CONTINUE
C
C       PROCESS GOES TO STATE J-1
C
        K=NSYS(IND)-1
        IND=IND+1
        NSYS(IND)=K
        P1(K+2)=P1(K+2)+T(IND)-T(IND-1)
        GO TO 20
   70   CONTINUE
        K=NSYS(IND)+1
        P1(K)=P1(K)+TSTOP-T(IND)
C
```

```
C      COMPUTE P2 THE DISCRETE TIME PROBABILITIES
C
       NSTOP=TSTOP+0.1
       JK=1
       DO 80 I=1,NSTOP
       DELT(I)=I
   75  CONTINUE
       JK=JK+1
       IF (DELT(I) .GE. T(JK)) GO TO 75
       JK=JK-1
       KK=NSYS(JK)+1
       P2(KK)=P2(KK)+1
   80  CONTINUE
       DO 90 I=1,M1
       X(I)=I-1
       P1(I)=P1(I)/TSTOP
       P2(I)=P2(I)/TSTOP
       P4(I)=P4(I)+P1(I)
       P5(I)=P5(I)+P2(I)
   90  CONTINUE
       IX=35641
   95  CONTINUE
       SP4=0.0
       SP5=0.0
       VP4=0.0
       VP5=0.0
C
C      CALCULATE STEADY STATE MEAN, VARIANCE AND
C      CUMULATIVE PROBABILITIES
C
       DO 99 I=1,M1
       P4(I)=P4(I)/NRUN
       P5(I)=P5(I)/NRUN
       SP4=SP4+P4(I)*(I-1)
       SP5=SP5+P5(I)*(I-1)
       VP4=VP4+P4(I)*(I-1)*(I-1)
       VP5=VP5+P5(I)*(I-1)*(I-1)
       IF (I .NE. 1) GO TO 96
       PDF4(I)=P4(I)
       PDF5(I)=P5(I)
       GO TO 97
   96  CONTINUE
       PDF4(I)=PDF4(I-1)+P4(I)
       PDF5(I)=PDF5(I-1)+P5(I)
   97  CONTINUE
       II=I-1
       WRITE (6,8200) II, P3(I), P4(I), P5(I), PDF4(I)
      1, PDF5(I)
 8200  FORMAT (I10,5F12.4,/)
   99  CONTINUE
C
C      PRINTOUT OF RESULTS
C
       WRITE (6,8800)
 8800  FORMAT (////)
       VP4=VP4-SP4**2
       VP5=VP5-SP5**2
       WRITE(6,7000) NRUN,SP4,VP4
 7000  FORMAT (10X,'TIME AVERAGE PROBABILITY WITH',I5,
      1 ' SIMULATION RUNS',/,10X,'MEAN = ',F10.4,'VARIANCE ='
      2 F10.4,////)
       WRITE(6,8000) NRUN,SP5,VP5
 8000  FORMAT (10X, 'DISCRETE TIME PROBABILITY WITH',I5,
      1 ' SIMULATION RUNS',/,10X,'MEAN = ',F10.4,
      2 '  VARIANCE =',F10.4,////)
       CALL PLOTP (X,P3,M1,0)
       WRITE (6,8800)
       CALL PLOTP (X,P4,M1,0)
       WRITE (6,8800)
       CALL PLOTP (X,P5,M1,0)
       STOP
       END
```

```
C
C
C
C      THIS PROGRAM CALCULATES THE STEADY STATE MEAN,VARIANCE
C      AND CUMULATIVE PROBABILITIES OF A BIRTH AND DEATH
C      PROCESS
C
C
C      M                   MAXIMUM NUMBER OF STATES
C      P                   VECTOR OF STEADY STATE PROBABILITIES
C      CL                  ARRIVAL RATE OF MESSAGES AT EACH MACHINE
C      CM                  DEPARTURE RATE OF A MESSAGE WHILE
C                          UNDERGOING TRANSMISSION
C      PDF                 CUMULATIVE STEADY STATE PROBABILITIES
C      XMEAN               STEADY STATE MEAN
C      XVAR                STEADY STATE VARIANCE
C
C
C
       DIMENSION X(201), P(201), PDF(201)
       READ(5,1000) NCARD
 1000  FORMAT (I5,2F10.4)
       DO 10 K=1,NCARD
       READ (5,1000) M, CL, CM
       WRITE(6,1000) M,CL,CM
       P(1)=1.0
       SUM =P(1)
       M1=M+1
       M2=2*M
       CL=CL/(M2-1)
       DO 20 I=2,M1
       J2=2*(I-2)
       C1=(M2-J2)*CL*(M2-J2-1)
       C2=(I-1)*CM
       P(I)=P(I-1)*C1/C2
       SCM=SUM+P(I)
   20  CONTINUE
       XMEAN=0
       XVAR=0.0
       WRITE (6,2000)
 2000  FORMAT (/////)
       DO 60 I=1,M1
       P(I)=P(I)/SUM
       X(I)=I-1
       XMEAN=XMEAN+P(I)*(I-1)
       XVAR=XVAR+P(I)*(I-1)*(I-1)
       IF (I .NE. 1) GO TO 30
       PDF(I)=P(I)
       GO TO 40
   30  CONTINUE
       PCF(I)=PDF(I-1)+P(I)
   40  CONTINUE
       II=I-1
       WRITE (6,3000) II, P(I), PDF(I)
 3000  FORMAT (I10,2F15.4,/)
   60  CONTINUE
       XVAR=XVAR-XMEAN**2
       WRITE (6,2000)
       WRITE (6,4000) XMEAN,XVAR
 4000  FORMAT (10X,'MEAN = ',F10.4,10X,'VARIANCE = '
      1, F10.4,////)
       CALL PLOTP (X,P,M1,0)
       WRITE (6,2000)
   10  CONTINUE
       STOP
       END
```

Listing of GPSS Program

This program simulates the arrivals and departures of messages at a system of 20 sendox machines. Functions EXP generates exponential variates with mean of one and UNI generates discrete uniform variates between 1 and 20. The program consists of 20 closely similar segments. Each segment simulates the activities at one machine. At the end of each simulation run, this program gives the expected waiting times at each machine and steady state probabilities of $X_M(t)$.

```
*
*
*
*       SIMULATION 'NO LOSS' MODEL
*       GPSS PROGRAM
*
*
*       EXP            GENERATE EXPONENTIAL VARIATE ,MEAN=1
*       UNI            GENERATE DISCRETE UNIFORM (1,20) VARIATE
*       SIZE           NOTE THE NUMBER OF BUSY MACHINES AT ONE
*                      MINUTE INTERVAL AND DIVIDE IT BY TWO
*       SERV           RANDOM TRANSMISSION TIME
*
*
*
EXP     FUNCTION   RN2,C24
0       0     .1   .104   .2    .222  .3    .355  .4    .509  .5    .69
.6      .915  .7   1.2    .75   1.38  .8    1.6   .84   1.83  .88   2.12
.9      2.3   .92  2.52   .94   2.81  .95   2.99  .96   3.2   .97   3.5
.98     3.9   .99  4.6    .995  5.3   .998  6.2   .999  7     .9997 8
UNI     FUNCTION   RN5,D20
.05     1     .1   2      .15   3     .2    4     .25   5     .3    6
.35     7     .4   8      .45   9     .5    10    .55   11    .6    12
.65     13    .7   14     .75   15    .8    16    .85   17    .9    18
.95     19    1.   20
1       VARIABLE   F1+F2+F3+F4+F5+F6+F7+F8+F9+F10
2       VARIABLE   F11+F12+F13+F14+F15+F16+F17+F18+F19+F20
SIZE    VARIABLE   (V1+V2)/2
SERV    FVARIABLE  FN$EXP*420
        SIMULATE
*
*       SIMULATE ARRIVAL OF INCOMING MESSAGES AT FACILITY 1
*
        GENERATE   1440,FN$EXP
*
*       TRY TO FIND A DISTINATION FACILITY NUMBER J
*       NOT EQUAL TO 1
TRY1    ASSIGN     1,FN$UNI
        TEST NE    P1,K1,TRY1
*
*       ENTER QUEUE
*
        QUEUE      1
        QUEUE      2
*
*       CHECK TO SEE WHETHER BOTH FACILITIES 1 & J
*       ARE AVAILABLE
ALL1    GATE NU    1
```

```
         GATE NU      P1
         TRANSFER     SIM,,ALL1
*
*        HOLD FACILITIES 1 & J FOR SERV SECONDS
*
         SEIZE        1
         SEIZE        P1
         DEPART       1
         ADVANCE      V$SERV
         RELEASE      P1
         RELEASE      1
         DEPART       2
*
*        END OF MESSAGE TRANSMISSION
*
         TERMINATE    1
         GENERATE     1440,FN$EXP
 TRY2    ASSIGN       1,FN$UNI
         TEST NE      P1,K2,TRY2
         QUEUE        3
         QUEUE        4
 ALL2    GATE NU      2
         GATE NU      P1
         TRANSFER     SIM,,ALL2
         SEIZE        2
         SEIZE        P1
         DEPART       3
         ADVANCE      V$SERV
         RELEASE      P1
         RELEASE      2
         DEPART       4
         TERMINATE    1
         GENERATE     1440,FN$EXP
 TRY3    ASSIGN       1,FN$UNI
         TEST NE      P1,K3,TRY3
         QUEUE        5
         QUEUE        6
 ALL3    GATE NU      3
         GATE NU      P1
         TRANSFER     SIM,,ALL3
         SEIZE        3
         SEIZE        P1
         DEPART       5
         ADVANCE      V$SERV
         RELEASE      P1
         RELEASE      3
         DEPART       6
         TERMINATE    1
```

```
         GENERATE    1440,FN$EXP
TRY4     ASSIGN      1,FN$UNI
         TEST NE     P1,K4,TRY4
         QUEUE       7
         QUEUE       8
ALL4     GATE NU     4
         GATE NU     P1
         TRANSFER    SIM,,ALL4
         SEIZE       4
         SEIZE       P1
         DEPART      7
         ADVANCE     V$SERV
         RELEASE     P1
         RELEASE     4
         DEPART      8
         TERMINATE   1
         GENERATE    1440,FN$EXP
TRY5     ASSIGN      1,FN$UNI
         TEST NE     P1,K5,TRY5
         QUEUE       9
         QUEUE       10
ALL5     GATE NU     5
         GATE NU     P1
         TRANSFER    SIM,,ALL5
         SEIZE       5
         SEIZE       P1
         DEPART      9
         ADVANCE     V$SERV
         RELEASE     P1
         RELEASE     5
         DEPART      10
         TERMINATE   1
         GENERATE    1440,FN$EXP
TRY6     ASSIGN      1,FN$UNI
         TEST NE     P1,K6,TRY6
         QUEUE       11
         QUEUE       12
ALL6     GATE NU     6
         GATE NU     P1
         TRANSFER    SIM,,ALL6
         SEIZE       6
         SEIZE       P1
         DEPART      11
         ADVANCE     V$SERV
         RELEASE     P1
         RELEASE     6
         DEPART      12
         TERMINATE   1
```

```
         GENERATE    1440,FN$EXP
TRY7     ASSIGN      1,FN$UNI
         TEST NE     P1,K7,TRY7
         QUEUE       13
         QUEUE       14
ALL7     GATE NU     7
         GATE NU     P1
         TRANSFER    SIM,,ALL7
         SEIZE       7
         SEIZE       P1
         DEPART      13
         ADVANCE     V$SERV
         RELEASE     P1
         RELEASE     7
         DEPART      14
         TERMINATE   1
         GENERATE    1440,FN$EXP
TRY8     ASSIGN      1,FN$UNI
         TEST NE     P1,K8,TRY8
         QUEUE       15
         QUEUE       16
ALL8     GATE NU     8
         GATE NU     P1
         TRANSFER    SIM,,ALL8
         SEIZE       8
         SEIZE       P1
         DEPART      15
         ADVANCE     V$SERV
         RELEASE     P1
         RELEASE     8
         DEPART      16
         TERMINATE   1
         GENERATE    1440,FN$EXP
TRY9     ASSIGN      1,FN$UNI
         TEST NE     P1,K9,TRY9
         QUEUE       17
         QUEUE       18
ALL9     GATE NU     9
         GATE NU     P1
         TRANSFER    SIM,,ALL9
         SEIZE       9
         SEIZE       P1
         DEPART      17
         ADVANCE     V$SERV
         RELEASE     P1
         RELEASE     9
         DEPART      18
         TERMINATE   1
```

```
      GENERATE    1440,FN$EXP
TRY10 ASSIGN      1,FN$UNI
      TEST NE     P1,K10,TRY10
      QUEUE       19
      QUEUE       20
ALL10 GATE NU     10
      GATE NU     P1
      TRANSFER    SIM,,ALL10
      SEIZE       10
      SEIZE       P1
      DEPART      19
      ADVANCE     V$SERV
      RELEASE     P1
      RELEASE     10
      DEPART      20
      TERMINATE   1
      GENERATE    1440,FN$EXP
TRY11 ASSIGN      1,FN$UNI
      TEST NE     P1,K11,TRY11
      QUEUE       21
      QUEUE       22
ALL11 GATE NU     11
      GATE NU     P1
      TRANSFER    SIM,,ALL11
      SEIZE       11
      SEIZE       P1
      DEPART      21
      ADVANCE     V$SERV
      RELEASE     P1
      RELEASE     11
      DEPART      22
      TERMINATE   1
      GENERATE    1440,FN$EXP
TRY12 ASSIGN      1,FN$UNI
      TEST NE     P1,K12,TRY12
      QUEUE       23
      QUEUE       24
ALL12 GATE NU     12
      GATE NU     P1
      TRANSFER    SIM,,ALL12
      SEIZE       12
      SEIZE       P1
      DEPART      23
      ADVANCE     V$SERV
      RELEASE     P1
      RELEASE     12
      DEPART      24
      TERMINATE   1
```

```
        GENERATE    1440,FN$EXP
TRY13   ASSIGN      1,FN$UNI
        TEST NE     P1,K13,TRY13
        QUEUE       25
        QUEUE       26
ALL13   GATE NU     13
        GATE NU     P1
        TRANSFER    SIM,,ALL13
        SEIZE       13
        SEIZE       P1
        DEPART      25
        ADVANCE     V$SERV
        RELEASE     P1
        RELEASE     13
        DEPART      26
        TERMINATE   1
        GENERATE    1440,FN$EXP
TRY14   ASSIGN      1,FN$UNI
        TEST NE     P1,K14,TRY14
        QUEUE       27
        QUEUE       28
ALL14   GATE NU     14
        GATE NU     P1
        TRANSFER    SIM,,ALL14
        SEIZE       14
        SEIZE       P1
        DEPART      27
        ADVANCE     V$SERV
        RELEASE     P1
        RELEASE     14
        DEPART      28
        TERMINATE   1
        GENERATE    1440,FN$EXP
TRY15   ASSIGN      1,FN$UNI
        TEST NE     P1,K15,TRY15
        QUEUE       29
        QUEUE       30
ALL15   GATE NU     15
        GATE NU     P1
        TRANSFER    SIM,,ALL15
        SEIZE       15
        SEIZE       P1
        DEPART      29
        ADVANCE     V$SERV
        RELEASE     P1
        RELEASE     15
        DEPART      30
        TERMINATE   1
```

```
         GENERATE    1440,FN$EXP
TRY16    ASSIGN      1,FN$UNI
         TEST NE     P1,K16,TRY16
         QUEUE       31
         QUEUE       32
ALL16    GATE NU     16
         GATE NU     P1
         TRANSFER    SIM,,ALL16
         SEIZE       16
         SEIZE       P1
         DEPART      31
         ADVANCE     V$SERV
         RELEASE     P1
         RELEASE     16
         DEPART      32
         TERMINATE   1
         GENERATE    1440,FN$EXP
TRY17    ASSIGN      1,FN$UNI
         TEST NE     P1,K17,TRY17
         QUEUE       33
         QUEUE       34
ALL17    GATE NU     17
         GATE NU     P1
         TRANSFER    SIM,,ALL17
         SEIZE       17
         SEIZE       P1
         DEPART      33
         ADVANCE     V$SERV
         RELEASE     P1
         RELEASE     17
         DEPART      34
         TERMINATE   1
         GENERATE    1440,FN$EXP
TRY18    ASSIGN      1,FN$UNI
         TEST NE     P1,K18,TRY18
         QUEUE       35
         QUEUE       36
ALL18    GATE NU     18
         GATE NU     P1
         TRANSFER    SIM,,ALL18
         SEIZE       18
         SEIZE       P1
         DEPART      35
         ADVANCE     V$SERV
         RELEASE     P1
         RELEASE     18
         DEPART      36
         TERMINATE   1
```

```
        GENERATE    1440,FN$EXP
TRY19   ASSIGN      1,FN$UNI
        TEST NE     P1,K19,TRY19
        QUEUE       37
        QUEUE       38
ALL19   GATE NU     19
        GATE NU     P1
        TRANSFER    SIM,,ALL19
        SEIZE       19
        SEIZE       P1
        DEPART      37
        ADVANCE     V$SERV
        RELEASE     P1
        RELEASE     19
        DEPART      38
        TERMINATE   1
        GENERATE    1440,FN$EXP
TRY20   ASSIGN      1,FN$UNI
        TEST NE     P1,K20,TRY20
        QUEUE       39
        QUEUE       40
ALL20   GATE NU     20
        GATE NU     P1
        TRANSFER    SIM,,ALL20
        SEIZE       20
        SEIZE       P1
        DEPART      39
        ADVANCE     V$SERV
        RELEASE     P1
        RELEASE     20
        DEPART      40
        TERMINATE   1
        GENERATE    60
        ASSIGN      1,V$SIZE
        TABULATE     1
        TERMINATE   0
1       TABLE       P1,0,1,12
        START       200
        RESET
        START       32000
        END
```

APPENDIX E

Calculation of λ'

In the previous sections, we had shown that a "no loss" model with input parameters N, λ and μ can be approximated by a "loss" model with input parameters N, λ' and μ, where λ' is chosen such that the effective arrival rate $\lambda'(1-y_e')^2$ for the "loss" model, is equal to λ.

When $X_M(t)$ is in steady state, the mean number of busy machines is Ny_e, hence the probability of finding a busy machine is Ny_e/N or y_e. In the "loss" model, a message is transmitted only if it finds both the transmitting and receiving machines free. The probability of this event is approximately $(1-y_e)^2$ assuming independence between availability of machines and a large N. Therefore, the effective arrival rate, λ_{eff}, for a "loss" model is $\lambda'(1-y_e')^2$ where y_e' is given by equation (4.15) as

$$y_e' = 1+\tfrac{1}{2}\rho' - \sqrt{(1+\tfrac{1}{2}\rho')^2-1}$$

with $\rho' = \frac{\mu}{2\lambda'}$

Therefore, we require that

$$\lambda'(\sqrt{(1+\tfrac{1}{2}\rho')^2-1} - \tfrac{1}{2}\rho')^2 = \lambda \qquad \text{(E.1)}$$

After expanding the quadratic term in equation (E.1) and rearranging terms, we obtain

$$(1+\tfrac{1}{2}\rho') - \frac{\lambda}{\lambda'\rho'} = \sqrt{\rho'(1+\tfrac{1}{4}\rho')} \qquad \text{(E.2)}$$

Squaring both sides of equation (E.2) and recalling that $\rho = \frac{\mu}{2\lambda}$, we obtain

$$\rho' = \rho + \frac{1}{\rho} - 2$$

Making the substitution for ρ and ρ', we finally have

$$\lambda' = \frac{\lambda}{(1 - \frac{1}{\rho})^2}$$

Using this rate of arrivals in our "loss" model we are able to approximate the system size characteristics of the "no loss" model with arrival rate λ.

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